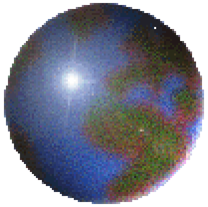
Linear Algebra



10.3 The Inverse Matrix

Multiplicative Inverse of a Matrix

If A is a square matrix of order n, then the inverse of matrix A, denoted by A^{-1} has the property

$$AA^{-1} = I_n, \quad A^{-1}A = I_n$$

where I_n is the identity matrix of order n.

The Inverse of a Matrix

- Inverse matrix:

Consider $A \in M_{n \times n}$

If there exists a matrix $B \in M_{n \times n}$ such that $AB = BA = I_n$,

, then (1) A is invertible (or nonsingular)

(2) B is the inverse of A

- **Note**:

A matrix that does not have an inverse is called noninvertible (or singular).

2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Properties

$$(I)\ (A^{-1})^{-1} = A$$

$$(II)\quad (AB)^{-1} = B^{-1}A^{-1}$$

- **Theorem**: The inverse of a matrix is unique

If B and C are both inverses of the matrix A, then $B = C$.

Pf:

$$AB = I$$

$$C(AB) = CI$$

$$(CA)B = C$$

$$IB = C$$

$$B = C$$

Consequently, the inverse of a matrix is unique.

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$\begin{bmatrix} A & | & I_n \end{bmatrix} \xrightarrow{\text{Use Elementary Row Operations}} \begin{bmatrix} I_n & | & A^{-1} \end{bmatrix}$$

Example: Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

Solution

$$\underset{A}{\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}} \vdots \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \xrightarrow[R_1+R_2;\,-4R_2+R_1]{} \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \vdots \underset{A^{-1}}{\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}}$$

If A can't be row reduced to I, then A is singular.

Example: Find the inverse of the following matrix

$$A=\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Solution

$$[A \ \vdots \ I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{aim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

If possible

$$\xrightarrow{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{6R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{4R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2}\begin{bmatrix} 1 & -1 & 0 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & -3 & -3 & -1 \\ 0 & 0 & 1 & \vdots & -2 & -4 & -1 \end{bmatrix} \xrightarrow{R_2+R_1}\begin{bmatrix} 1 & 0 & 0 & \vdots & -2 & -3 & -1 \\ 0 & 1 & 0 & \vdots & -3 & -3 & -1 \\ 0 & 0 & 1 & \vdots & -1 & -4 & -1 \end{bmatrix}$$

$$= [\, I \;\vdots\; A^{-1} \,]$$

So the matrix A is invertible, and its inverse is

$$A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$$

- Check:

$$AA^{-1} = A^{-1}A = I$$

Example: Show that the matrix $\begin{bmatrix} 1 & -6 & 4 \\ 3 & 4 & 2 \\ 5 & 3 & 5 \end{bmatrix}$ is a singular matrix.

Solution

$$\left[\begin{array}{ccc|ccc} 1 & -6 & 4 & 1 & 0 & 0 \\ 3 & 4 & 2 & 0 & 1 & 0 \\ 5 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{aim(if possible)}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right]$$

$$
\left.\begin{array}{l}
-3R_1 + R_2 \\
-5R_1 + R_3 \\
\dfrac{1}{22}R_2 \\
-33R_2 + R
\end{array}\right\}
\xrightarrow{\text{produces the matrix}}
\left[\begin{array}{ccc|ccc}
1 & -6 & 4 & 1 & 0 & 1 \\
0 & 1 & -\dfrac{5}{11} & \dfrac{-3}{22} & \dfrac{1}{22} & 0 \\
0 & 0 & 0 & \dfrac{-1}{2} & -\dfrac{3}{2} & 1
\end{array}\right]
$$

Because
a row of zeros came out, the matrix is singular(no inverse)

- Power of a square matrix:

(1) $A^0 = I$

(2) $A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$

(3) $A^r \cdot A^s = A^{r+s} \qquad r, s : \text{integers}$

$(A^r)^s = A^{rs}$

(4) $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$

- **Theorem**： Properties of inverse matrices

If A is an invertible matrix, k is a positive integer, and c is a scalar, then

$$\text{(1) } A^{-1} \text{ is invertible and } (A^{-1})^{-1} = A$$

$$\text{(2) } A^{k} \text{ is invertible and } (A^{k})^{-1} = A^{-1*k} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^{k} = A^{-k}$$

$$\text{(3) } cA \text{ is invertible and } (cA)^{-1} = \frac{1}{c}A^{-1},\ c \neq 0$$

- **Theorem:** The inverse of a product

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I)A^{-1} = (AI)A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}(I)B = B^{-1}(IB) = B^{-1}B = I$$

and since the inverse is unique.

So $(AB)^{-1} = B^{-1}A^{-1}$

- **Note:**

$$\left(A_1A_2A_3\cdots A_n\right)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- **Theorem:** Cancellation properties

  If **C is an invertible matrix**, then the following properties hold:

  (1) If $AC=BC$, then $A=B$ (Right cancellation property)

  (2) If $CA=CB$, then $A=B$ (Left cancellation property)

Pf: $$AC = BC$$

$$(AC)C^{-1} = (BC)C^{-1}$$ (C is invertible, so C^{-1} exists)

$$A(CC^{-1}) = B(CC^{-1})$$

$$AI = BI$$

$$A = B$$

Note:

If C is not invertible, then cancellation is not valid.

- **Theorem:** Systems of linear equations with unique solutions

If $\boldsymbol{A}$ is an invertible matrix, then the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution given by $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

Pf:

$$Ax = b$$

$$A^{-1}Ax = A^{-1}b \qquad (A \text{ is nonsingular})$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

If x_1 and x_2 were two solutions of equation $Ax = b$.

then $Ax_1 = b = Ax_2 \Rightarrow x_1 = x_2$ (Left cancellation property)

This solution is unique.

- **Note:**

$$Ax = b$$

$$\begin{bmatrix} A & \mid & b \end{bmatrix} \xrightarrow{A^{-1}} \begin{bmatrix} A^{-1}A & \mid & A^{-1}b \end{bmatrix} = \begin{bmatrix} I & \mid & A^{-1}b \end{bmatrix}$$

- Example: Solve the following system using the inverse

3x - 2y + z = 9
x + 2y - 2z = -5
x + y - 4z = -2

Solution

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \\ -2 \end{bmatrix}$$

Matrix Form

$$\begin{bmatrix} 3 & -2 & 1 \\ 1 & 2 & -2 \\ 1 & 1 & -4 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{6}{23} & \frac{7}{23} & -\frac{2}{23} \\ -\frac{2}{23} & \frac{13}{23} & -\frac{7}{23} \\ \frac{1}{23} & \frac{5}{23} & -\frac{8}{23} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6}{23} & \frac{7}{23} & -\frac{2}{23} \\ -\frac{2}{23} & \frac{13}{23} & -\frac{7}{23} \\ \frac{1}{23} & \frac{5}{23} & -\frac{8}{23} \end{bmatrix} \bullet \begin{bmatrix} 9 \\ -5 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$