8.3 THE HYPERBOLA

OBJECTIVES

1. Define a Hyperbola
2. Find the Standard Form of the Equation of a Hyperbola
3. Find the Transverse Axis
4. Find the Eccentricity of a Hyperbola
5. Find the Asymptotes of a Hyperbola
6. Graph a Hyperbola

• HPERBOLAS WITH CENTER AT ORIGIN $(0,0)$

It was shown in the introduction of chapter 8, that a hyperbola is the curve that occurs at the intersection of a cone with two nappes and a plane.

The definition of a hyperbola is similar to that of an ellipse. The only change is that instead of using the sum of distances from two fixed points, we use the difference.

Definition of an Hyperbola

A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points(the **foci**) in the plane is a **positive constant**

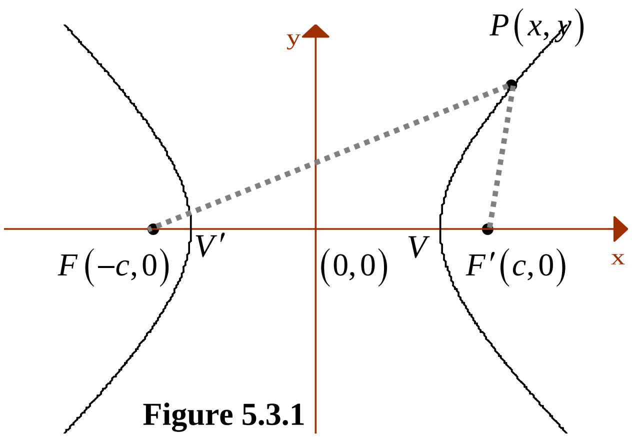

Figure 5.3.1

To obtain a simple equation for a hyperbola, let us choose the foci to be at $F(c,0)$ and $F'(-c,0)$ where $c>0$, and denote the constant distance by $2a$ where $a>0$. The midpoint of the line segment FF'(the origin$(0,0)$) is called the **center** of the hyperbola. According to the definition of a hyperbola, if the point $P(x,y)$ is on the hyperbola (see Figure 5.3.1), then we must have

$$\left|\sqrt{(x-c)^2+y^2}-\sqrt{(x+c)^2+y^2}\right|=2a$$

From which we get

$$\frac{x^2}{a^2}-\frac{y^2}{c^2-a^2}=1 \quad \textbf{(see the textbook)}$$

$$where \ \ c>a$$
$$\Rightarrow c^2-a^2>0$$

Since c^2-a^2 is positive, we may replace it by another positive number, b^2. Thus

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \text{ where } b^2=c^2-a^2$$

is the **standard form** of the equation of a hyperbola centered at the origin with foci on the $x-$axis.

Hyperbola whose Transverse Axis in on the x-axis:

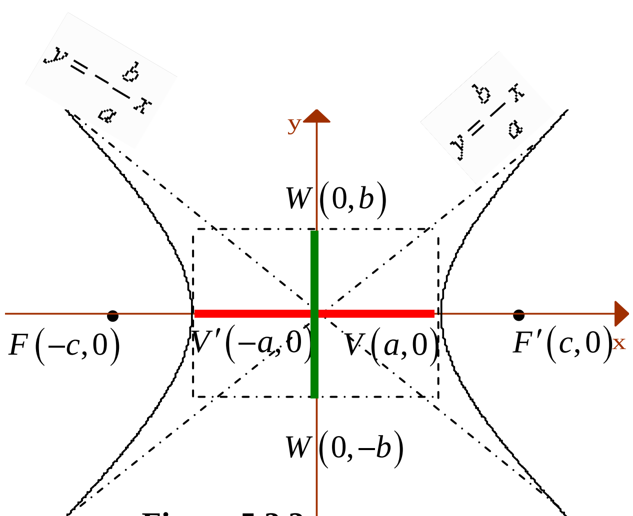

Figure 5.3.2

Properties:

- ✓ Applying tests for symmetry, we see that the hyperbola is symmetric with respect to both axes and the origin(Why?).
- ✓ Letting $y=0$ in the standard form gives $x^2=a^2 \Rightarrow x=\pm a$

thus the x-intercepts of hyperbola are $V(a,0)$ and $V'(-a,0)$ as shown in Figure 5.3.2 .

- ✓ V and V' are called the **vertices** of the hyperbola. The line segment VV' is called the **transverse axis** of the hyperbola.
- ✓ The graph has no y-intercepts, since the eq
- ✓ uation $\dfrac{-y^2}{b^2}=1$ has the nonreal solutions $y=\pm ib$.
- ✓ However, the points $W(0,b)$ and $W'(0,-b)$ are very important, as we will see. The line segment WW' is called the **conjugate axis** of the hyperbola.

Now, as you see in Figure 5.3.1 and Figure 5.3.2, that the graph of a hyperbola has two branches. The two branches looks like parabolas, but they are not parabolas. The branches of the hyperbola shown in Figure 5.3.2 get closer and closer to the dashed lines, called the **asymptotes,** but they never intersect them. The asymptotes are used as guidelines in sketching hyperbola. The asymptotes are found by extending the diagonals of the **fundamental rectangle,** shown in Figure 5.3.2.

It is clear from Figure 5.3.2. that the asymptote with positive slope passes through origin and the point (a,b) while the other asymptote passes through origin and the point $(-a,b)$.

Therefore the lines

$$y=\frac{b}{a}x \text{ and } y=\frac{-b}{a}x$$

are asymptotes for the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

EXAMPLE 1

Sketch and describe the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Solution

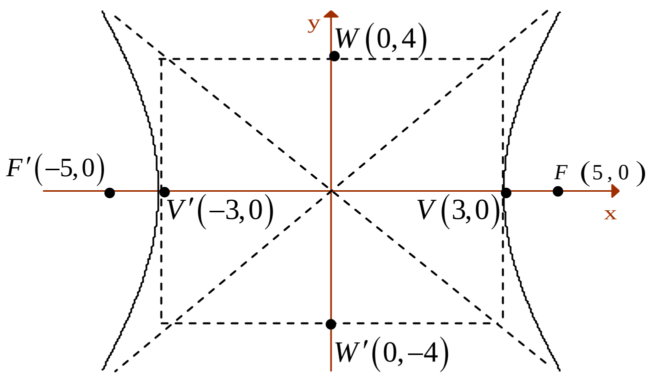

Figure 5.3.3

Compare $\frac{x^2}{9} - \frac{y^2}{16} = 1$ with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we see that $a^2 = 9$ and $b^2 = 16$

Hence $c^2 = a^2 + b^2 = 25.$

Thus the equation represents a hyperbola with the following properties:

- Vertices at $V(3,0)$ and $V'(-3,0)$
- Center at (0,0)
- Transverse axis on the x-axis.
- Conjugate axis on the y-axis with endpoints $W(0,4)$ and $W'(0,-4)$.
- Foci at $F(5,0)$ and $F'(-5,0)$
- The equations of the asymptotes are $y = \frac{4}{3}x$ and $y = -\frac{4}{3}x$.

Now draw the fundamental rectangle whose vertices are at $(\pm 3, 4)$ and $(3, \pm 4)$ whose diagonal lie on the asymptotes of the hyperbola as shown in Figure 5.3.3. A rough sketch of the hyperbola can be drawn as in Figure 5.3.3.

Hyperbola with Transverse Axis on the y-axis

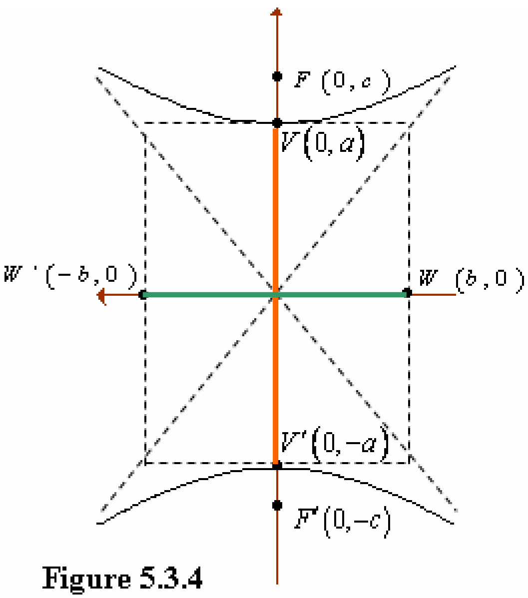

Figure 5.3.4

Similarly, it can be shown that if we take the foci of the hyperbola on the y-axis, we obtain the equation

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

with the following properties:

- Vertices at $V(0, a)$ and $V'(0, -a)$
- Transverse axis on the y-axis with endpoints $V(0, a)$ and $V'(0, -a)$
- **Conjugate** axis on the x-axis with endpoints $W(b, 0)$ and $W'(-b, 0)$
- Foci at $F(0, c)$ and $F'(0, -c)$ where $c^2 = a^2 + b^2$
- The equation of the asymptotes are $y = \frac{a}{b}x$ and $y = \frac{-a}{b}x$ as shown in Figure 5.3.4

EXAMPLE 2

Sketch and describe the hyperbola $\frac{y^2}{9}-\frac{x^2}{16}=1.$

Solution

If we follow the same steps as in Example 1, we get the properties of this hyperbola as follows, and its graph as shown in Figure 5.3.5

Properties of the given hyperbola

- The transverse axis lies on the $y-$axis (Why?) and with center at origin.
- $a^2=9 \Rightarrow a=3 \Rightarrow$ vertices at $V(0,3)$ and $V'(0,-3)$
- $b^2=16 \Rightarrow b=4 \Rightarrow$ the endpoints of the conjugate axis are at $W(4,0)$ and $W'(-4,0)$
- $c^2=a^2+b^2=25 \Rightarrow c=5 \Rightarrow$ foci at $F(0,5)$ and $F'(0,-5)$
- The equation of the asymptotes are $y=\frac{3}{4}x$ and $y=-\frac{3}{4}x$

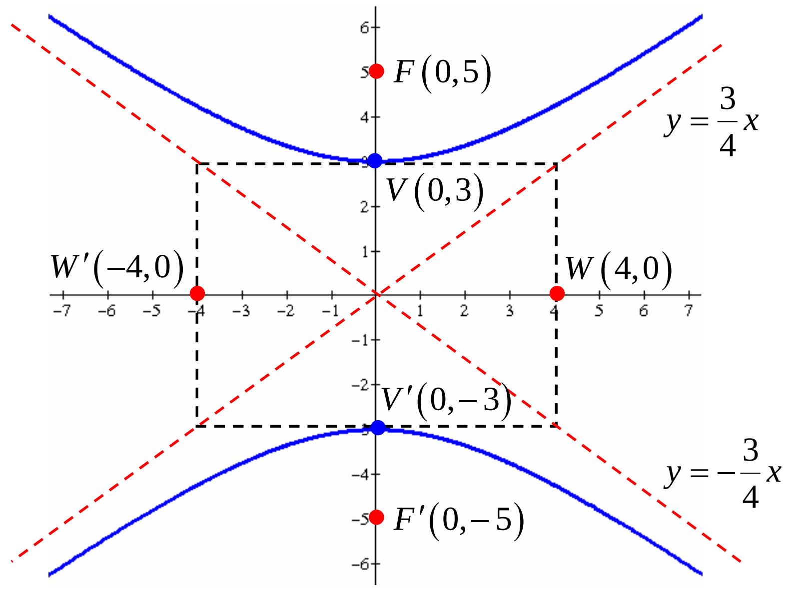

Figure 5.3.5

• THE ECCENTRICITY OF A HYPERBOLA

As in the case of the ellipse, the eccentricity e of a hyperbola is given by $e = \frac{c}{a}$

Since $c > a$, then $e > 1$.

Actually, the graph of a hyperbola can be very wide or very narrow:

- ❖ narrow hyperbolas have e near 1
- ❖ wide hyperbolas has large e as shown in Figure 5.3.6

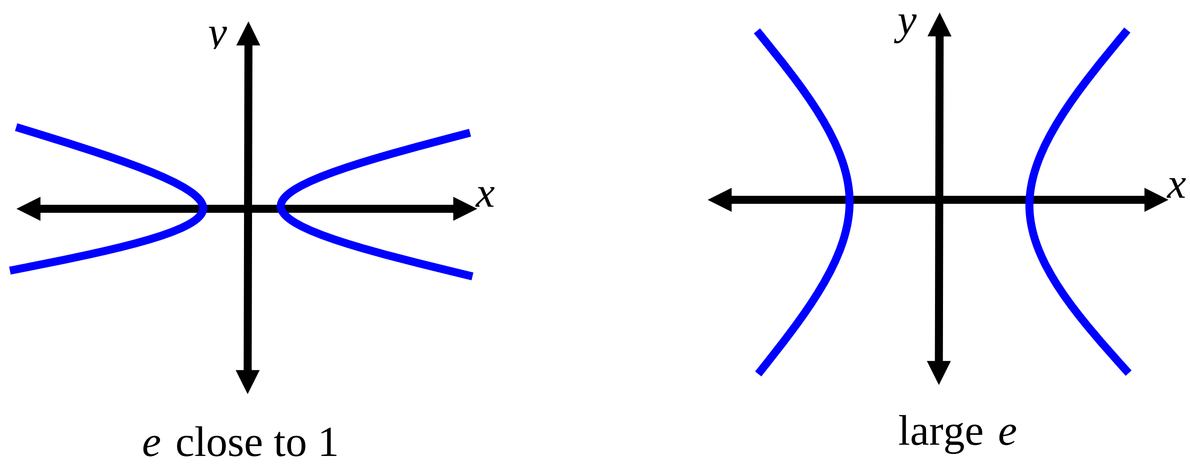

Figure 5.3.6

EXAMPLE 3

The hyperbolas $\frac{x^2}{9}-\frac{y^2}{16}=1$, $\frac{y^2}{9}-\frac{x^2}{16}=1$ given in examples 1 and 2 have the same eccentricity $e=\frac{c}{a}=\frac{5}{3}$.

- **HYPERBOLA WITH CENTER AT (h,k)**

Like circles and ellipses, hyperbolas may be centered at any point in the plane. To get the equation of a hyperbola centered at (h,k), replace x by $x-h$ and y by $y-k$ in the equation of the hyperbola centered at origin as shown in the following tables and Figures.

In the two cases that are given below:

- Move a units to the right to find V and a units to the left to find V' (endpoints of the transverse axis)
- Move c units to the right to find F and c units to the left to find F'
- Move b units to up and b units to down to find the endpoints of the conjugate axis.

Transverse Axis Parallel to the $x-$axis

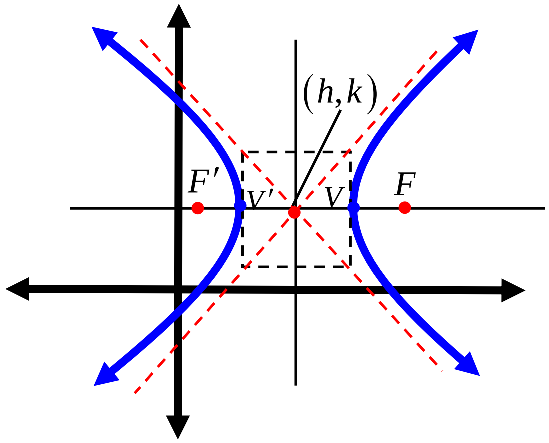

Figure 5.3.7

- Equation:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

- Vertices:

$V(h+a,k)$ and $V'(h-a,k)$

- Foci:

$F(h+c,k)$ and $F'(h-c,k)$,

where $c^2=a^2+b^2$

- Asymptotes:

$$y-k=\pm\frac{b}{a}(x-h)$$

- Eccentricity:

$$e=\frac{c}{a}$$

Observe that the vertices and foci lie on the line $y=k$ while the conjugate axis lies on the line $x=h$

Transverse Axis Parallel to the $y-$axis

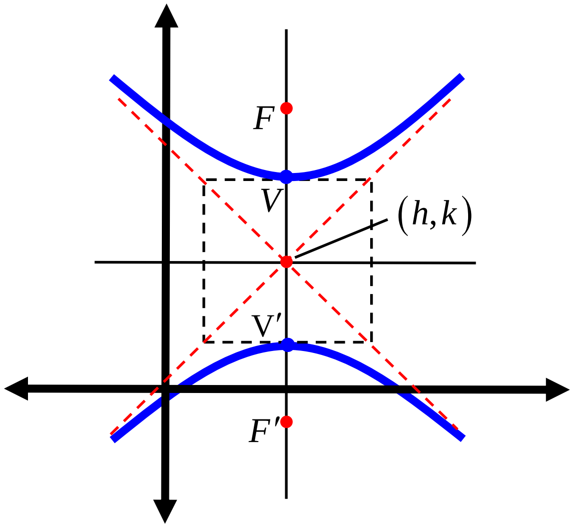

Figure 5.3.7

- Equation:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

- Vertices:

$V(h,k+a)$ and $V'(h,k-a)$

- Foci:

$F(h,k+c)$ and $F'(h,k-c)$,

where $c^2=a^2+b^2$

- Asymptotes:

$$y-k=\pm\frac{a}{b}(x-h)$$

- Eccentricity:

$$e=\frac{c}{a}$$

In this case the vertices and foci lie on the line $x=h$ while the conjugate axis lies on the line $y=k$

EXAMPLE 4

Show that the equation $4x^2 - y^2 + 16x + 8y - 16 = 0$ represents a hyperbola. Then describe the hyperbola and sketch its graph.

Solution

The given equation can be rewritten as:

$$4(x^2 + 4x) - (y^2 - 8y) = 16$$

We now complete the squares:

$$4(x^2 + 4x + 4) - (y^2 - 8y + 16) = 16 + 16 - 16$$

$$\Rightarrow 4(x+2)^2 - (y-4)^2 = 16$$

$$\Rightarrow \frac{(x+2)^2}{4} - \frac{(y-4)^2}{16} = 1$$

Which is the standard form of a hyperbola with the following properties:

- center at $(-2,4)$
- Transverse axis is parallel to the $x-$axis and lies on the line $y = 4$
- $a^2 = 4 \Rightarrow a = 2 \Rightarrow$ vertices at $V(-2+2,4) = (0,4)$ and $V'(-2-2,4) = (-4,4)$
- $b^2 = 16 \Rightarrow b = 4 \Rightarrow$ endpoints of the conjugate axis $W(-2,4+4) = (-2,8)$ and $W'(-2,4-4) = (-2,0)$, and lies on the line $x = -2$

- $c^2 = a^2 + b^2 = 20 \Rightarrow c = 2\sqrt{5} \Rightarrow$ foci at $F\left(-2+2\sqrt{5},4\right)$ and $F'\left(-2-2\sqrt{5},4\right)$, and lies on the line $y = 4$
- Asymptotes: $y-k = \pm\dfrac{a}{b}(x-h) \Rightarrow y-4 = 2(x+2)$ and $y-4 = -(x+2)$
- Eccentricity $e = \dfrac{c}{a} \Rightarrow e = \sqrt{5}$
- The graph of this hyperbola is shown in Figure 5.3.8

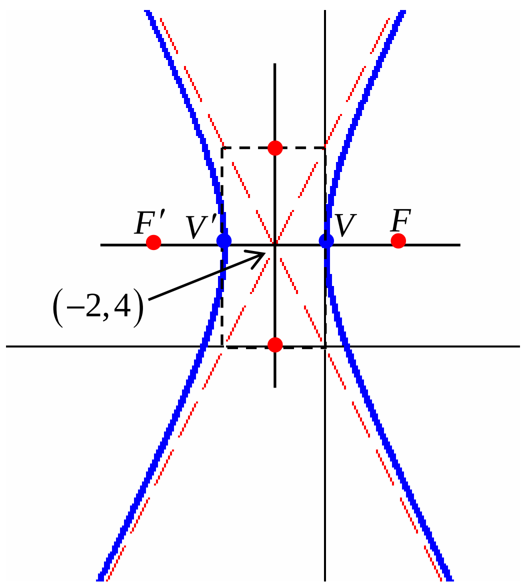

Figure 5.3.8