Linear Algebra

10.4 Determinants

Definition

The **determinant** of a 2 × 2 matrix *A* is denoted **|*A*|** and is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = +a_{11}a_{22} - a_{12}a_{21}$$

$-$ $+$

Observe that the determinant of a 2 × 2 matrix is given by *the different of the products of the two diagonals* of the matrix.

The notation **det(*A*)** is also used for the determinant of *A*.

Example 1

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2 \times 1) - (4 \times (-3)) = 2 + 12 = 14$$

To define the determinant of a matrix of order greater than 2, we first need two other definitions

Definitions:

(I) Let A be a square matrix.

The **minor** of the element a_{ij} is denoted M_{ij} and is the determinant of the matrix that remains after deleting row i and column j of A.

(II) The **cofactor** of a_{ij} is denoted C_{ij} and is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Note that $C_{ij} = M_{ij}$ *or* $-M_{ij}$.

Example 2

Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix *A*.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$$

Solution

$$\text{Minor of } a_{11}: \ M_{11} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = (-1\times1)-(2\times(-2)) = 3$$

$$\text{Cofactor of } a_{11}: \ C_{11} = (-1)^{1+1} M_{11} = (-1)^2(3) = 3$$

$$\text{Minor of } a_{32}: \ M_{32} = \begin{vmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = (1\times2)-(3\times4) = -10$$

$$\text{Cofactor of } a_{32}: \ C_{32} = (-1)^{3+2} M_{32} = (-1)^5(-10) = 10$$

Theorem

The determinant of a square matrix is the sum of the products of the elements of any row or column and their cofactors.

ith row expansion: $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

jth column expansion: $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

Example 4

It is advised to use the row(column) containing the most number of zeros, this reduces the number of calculations.

Find the determinant of the following matrix using the second row.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Solution

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= -3\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$$

$$= -3[(2\times 1) - (-1\times 2)] + 0[(1\times 1) - (-1\times 4)] - 1[(1\times 2) - (2\times 4)]$$

$$= -12 + 0 + 6 = -6$$

Or alternatively, you may use row one as shown beloe:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\ &= [(0\times 1) - (1\times 2)] - 2[(3\times 1) - (1\times 4)] - [(3\times 2) - (0\times 4)] \\ &= -2 + 2 - 6 \\ &= -6 \end{aligned}$$

which is the same result that was obtained using row two.

Example 5

Evaluate the determinant of the following 4 × 4 matrix.

$$\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$$

Solution

$$|A| = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$$

$$= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43})$$

$$= 3\begin{vmatrix} 2 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= 3(2)\begin{vmatrix} -1 & 2 \\ 1 & -3 \end{vmatrix} = 6(3-2) = 6$$

Example 6

Solve the following equation for the variable *x*.

$$\begin{vmatrix} x & x+1 \\ -1 & x-2 \end{vmatrix} = 7$$

Solution

Expand the determinant to get the equation

$$x(x-2)-(x+1)(-1)=7$$

Proceed to simplify this equation and solve for *x*.

$$x^2-2x+x+1=7$$
$$x^2-x-6=0$$
$$(x+2)(x-3)=0$$
$$x=-2 \text{ or } 3$$

There are two solutions to this equation, $x = -2$ or 3.

Short Cuts to Computing Determinants of 2 × 2 and 3 × 3 Matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = a_{11}a_{22} - a_{12}a_{21}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right]$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

Properties of Determinants

Theorem

Let A be an $n \times n$ matrix and c be a nonzero scalar.

(a) If $A \underset{cR_i}{\approx} B$ then $|B| = c|A|$. (Multiplying a row by a nonzero number)

(b) If $A \underset{Ri \leftrightarrow Rj}{\approx} B$ then $|B| = -|A|$. (Switching two rows)

(c) If $A \underset{Ri+cRj}{\approx} B$ then $|B| = |A|$. (Adding a row to a multiple of another row)

Proof (a)

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \dots + a_{kn}C_{kn}$$

$$|B| = ca_{k1}C_{k1} + ca_{k2}C_{k2} + \dots + ca_{kn}C_{kn}$$

$$\therefore |B| = c|A|.$$

Example 7

Evaluate the determinant $\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}$.

Solution

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \underset{C2+2C3}{=} \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix} = (-3)\begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = -21$$

Example 8

If $A=\begin{bmatrix}1 & 4 & 3\\ 0 & 2 & 5\\ -2 & -4 & 10\end{bmatrix}$, $|A| = 12$ is known.

Evaluate the determinants of the following matrices.

(a) $B_1=\begin{bmatrix}1 & 12 & 3\\ 0 & 6 & 5\\ -2 & -12 & 10\end{bmatrix}$ (b) $B_2=\begin{bmatrix}1 & 4 & 3\\ -2 & -4 & 10\\ 0 & 2 & 5\end{bmatrix}$ (c) $B_3=\begin{bmatrix}1 & 4 & 3\\ 0 & 2 & 5\\ 0 & 4 & 16\end{bmatrix}$

Solution

(a) $A \underset{3C2}{\approx} B_1$ Thus $|B_1| = 3|A| = 36$.

(b) $A \underset{R2\leftrightarrow R3}{\approx} B_2$ Thus $|B_2| = -|A| = -12$.

(c) $A \underset{R3+2R1}{\approx} B_3$ Thus $|B_3| = |A| = 12$.

Definition

A square matrix A is said to be **singular** if $|A|=0$.

A is **nonsingular** if $|A|\neq 0$.

Theorem

Let A be a square matrix. $|A|=0$ (A is singular if

(a) all the elements of a row (column) are zero.

(b) two rows (columns) are equal.

(c) two rows (columns) are proportional. (i.e., $Ri=cRj$)

Proof

(a) Let all elements of the kth row of A be zero.

$$|A| = a_{k1}C_{k1} + a_{k2}C_{k2} + \cdots + a_{kn}C_{kn} = 0C_{k1} + 0C_{k2} + \cdots + 0C_{kn} = 0$$

(c) If $Ri=cRj$, then $A \underset{Ri-cRj}{\approx} B$, row i of B is [0 0 … 0].

$\Rightarrow |A|=|B|=0$

Example 9

Show that the following matrices are singular.

$$\text{(a) } A = \begin{bmatrix} 2 & 0 & -7 \\ 3 & 0 & 1 \\ -4 & 0 & 9 \end{bmatrix} \quad \text{(b) } B = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$$

Solution

(a) All the elements in column 2 of A are zero. Thus $|A| = 0$.

(b) Row 2 and row 3 are proportional. Thus $|B| = 0$.

Theorem

Let A and B be $n \times n$ matrices and c be a nonzero scalar.

(a) $|cA| = c^n|A|$.

(b) $|AB| = |A||B|$.

(d) $\left|A^{-1}\right| = \dfrac{1}{|A|}$ (assuming A^{-1} exists)

Proof

(a)

$$A \underset{cR1,\, cR2,\, \ldots,\, cRn}{\approx} cA \quad \Rightarrow \quad |cA| = c^n |A|$$

(d)

$$|A| \cdot \left|A^{-1}\right| = \left|A \cdot A^{-1}\right| = |I| = 1 \quad \Rightarrow \quad \left|A^{-1}\right| = \frac{1}{|A|}$$

Example 10

If A is a 2×2 matrix with $|A| = 4$, use Theorem 3.4 to compute the following determinants.

(a) $|3A|$ (b) $|A^2|$ (c) $|5A^{-1}|$, assuming A^{-1} exists

Solution

(a) $|3A| = (3^2)|A| = 9 \times 4 = 36.$

(b) $|A^2| = |AA| = |A|\,|A| = 4 \times 4 = 16.$

(c) $|5A^{-1}| = (5^2)|A^{-1}| = 25|A^{-1}| = 25\frac{1}{|A|} = \frac{25}{4}.$

Example 11

Prove that if A and B are square matrices of the same size, with A being singular, then AB is also singular. Is the converse true?

Solution

$(\Rightarrow)$

$$|A| = 0 \quad \Rightarrow \quad |AB| = |A||B| = 0$$

Thus the matrix AB is singular.

$(\Leftarrow)$

$$|AB| = 0 \Rightarrow |A||B| = 0 \Rightarrow |A| = 0 \text{ or } |B| = 0$$

Thus AB being singular implies that either A or B is singular.

The inverse is not true.

Homework

Exercise 11

Prove the following identity without evaluating the determinants.

$$\begin{vmatrix} a+b & c+d & e+f \\ p & q & r \\ u & v & w \end{vmatrix} = \begin{vmatrix} a & c & e \\ p & q & r \\ u & v & w \end{vmatrix} + \begin{vmatrix} b & d & f \\ p & q & r \\ u & v & w \end{vmatrix}$$

Hint: try

$$\begin{vmatrix} a+b & c+d & e+f \\ p & q & r \\ u & v & w \end{vmatrix} = (a+b)\begin{vmatrix} q & r \\ v & w \end{vmatrix} - (c+d)\begin{vmatrix} p & r \\ u & w \end{vmatrix} + (e+f)\begin{vmatrix} p & q \\ u & v \end{vmatrix}$$

Numerical Evaluation of a Determinant

Definition

A square matrix is called an **upper triangular matrix** if all the elements below the main diagonal are zero.

It is called a **lower triangular matrix** if all the elements above the main diagonal are zero.

$$\begin{bmatrix} 3 & 8 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 0 & 7 \\ 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 7 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 9 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 7 & 0 & 2 & 0 \\ 4 & 5 & 8 & 1 \end{bmatrix}$$

upper – triangular   lower – triangular

Theorem

The determinant of a triangular matrix is the product of its diagonal elements.

Proof

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = \cdots = a_{11}a_{22}\cdots a_{nn}$$

Example 12

Let $A = \begin{bmatrix} 2 & -1 & 9 \\ 0 & 3 & -4 \\ 0 & 0 & -5 \end{bmatrix}$, find $|A|$.

Sol. $|A| = 2 \times 3 \times (-5) = -30.$

Example 13

Evaluation the determinant. $\begin{bmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{bmatrix}$

Solution elementary row operations

$$\begin{vmatrix} 2 & 4 & 1 \\ -2 & -5 & 4 \\ 4 & 9 & 10 \end{vmatrix} \begin{matrix} = \\ R2+R1 \\ R3+(-2)R1 \end{matrix} \begin{vmatrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 1 & 8 \end{vmatrix}$$

$$\begin{matrix} = \\ R3+R2 \end{matrix} \begin{vmatrix} 2 & 4 & 1 \\ 0 & -1 & 5 \\ 0 & 0 & 13 \end{vmatrix} = 2\times(-1)\times 13 = -26$$

Example 14

Evaluation the determinant. $\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix} \begin{matrix} = \\ R2+(-2)R1 \\ R3+(-1)R1 \\ R4+R1 \end{matrix} \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 4 & 2 \end{vmatrix}$$

$$\begin{matrix} = \\ R4+2R3 \end{matrix} \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 6 \end{vmatrix}$$

$$= 1 \times (-1) \times (-2) \times 6 = 12$$

Example 15

Evaluation the determinant. $\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & -2 & 4 \\ -1 & 2 & -5 \\ 2 & -2 & 11 \end{vmatrix} \underset{\substack{R2+R1 \\ R3+(-1)R1}}{=} \begin{vmatrix} 1 & -2 & 4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{vmatrix}$$

$$\underset{R2 \leftrightarrow R3}{=} (-1)\begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{vmatrix}$$

$$= (-1)\times 1 \times 2 \times (-1) = 2$$

Example 16

Evaluation the determinant. $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$

Solution

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix} \begin{matrix} = \\ R2+R1 \\ R3+(-2)R1 \\ R4+(-6)R1 \end{matrix} \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & -11 \end{vmatrix} = 0$$

diagonal element is zero and all elements below this diagonal element are zero.

Theorem

A square matrix A is invertible if and only if $|A| \neq 0$.

Proof

($\Rightarrow$) Assume that A is invertible.

$$\Rightarrow \; AA^{-1} = I_n.$$

$$\Rightarrow \; |AA^{-1}| = |I_n|.$$

$$\Rightarrow \; |A||A^{-1}| = 1$$

$$\Rightarrow \; |A| \neq 0.$$

($\Leftarrow$) Theorem 3.6 tells us that if $|A| \neq 0$, then A is invertible.

$$A^{-1} \textit{ exists if and only if } |A| \neq 0.$$

Example 17

Use a determinant to find out which of the following matrices are invertible.

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 4 & -3 \\ 4 & 12 & -7 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{bmatrix}$$

Solution

$|A| = 5 \neq 0$. *A* is invertible.

$|B| = 0$. *B* is singular. The inverse does not exist.

$|C| = 0$. C is singular. The inverse does not exist.

$|D| = 2 \neq 0$. *D* is invertible.

Theorem

Let $AX = B$ be a system of n linear equations in n variables.

(1) If $|A| \neq 0$, there is a unique solution.

(2) If $|A| = 0$, there may be many or no solutions.

Proof

(1) If $|A| \neq 0$

$\Rightarrow A^{-1}$ exists

$\Rightarrow$ there is then a unique solution given by $X = A^{-1}B$ (Thm 2.9).

(2) If $|A| = 0$

$\Rightarrow$ since $A \approx \ldots \approx C$ implies that if $|A| \neq 0$ then $|C| \neq 0$ (Thm 3.2).

$\Rightarrow$ the reduced echelon form of A is not I_n.

$\Rightarrow$ The solution to the system $AX = B$ is not unique.

$\Rightarrow$ many or no solutions.

Example 18

Determine whether or not the following system of equations has an unique solution.

$$\begin{aligned} 3x_1 + 3x_2 - 2x_3 &= 2 \\ 4x_1 + x_2 + 3x_3 &= -5 \\ 7x_1 + 4x_2 + x_3 &= 9 \end{aligned}$$

Solution

Since

$$\begin{vmatrix} 3 & 3 & -2 \\ 4 & 1 & 3 \\ 7 & 4 & 1 \end{vmatrix} = 0$$

Thus the system does not have an unique solution.

Homework

Exercise

Show that if $A = A^{-1}$, then $|A| = \pm 1$.