10.2 The Algebra of Matrices

Objectives

- Operations on Matrices: Sums, Scalar Products;
- Matrix Multiplication; Mechanics of Matrix Multiplication

Operations on Matrices

- Equal matrices:

$$\textbf{Let } A = [a_{ij}]_{m\times n}, \quad B = [b_{ij}]_{m\times n}$$

$$A = B \quad \textbf{if and only if} \quad a_{ij} = b_{ij} \quad \forall\ 1 \le i \le m,\ \ 1 \le j \le n$$

- Ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\text{If} \quad A = B$$

$$\text{Then} \quad a = 1, \quad b = 2, \quad c = 3, \quad d = 4$$

Definition The *sum* of two same-sized matrices is their entry-by-entry sum. The *scalar multiple* of a matrix is the result of entry-by-entry scalar multiplication.

- Matrix addition:

If $A=[a_{ij}]_{m\times n},\quad B=[b_{ij}]_{m\times n}$

Then $A+B=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

- Scalar Multiplication:

If $A=[a_{ij}]_{m\times n},\quad c:\text{scalar}$

Then $cA=[ca_{ij}]_{m\times n}$

- Matrix subtraction:

$$A - B = A + (-1)B$$

- Example: Matrix addition

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Matrix Multiplication

- Matrix multiplication of matrices : A_{mxk} and B_{kxn}

➢ number of columns of the first must equal number of rows of the second

the **product** is a matrix, denoted AB = C

The *i*, *j* entry of the matrix product is the dot product of row *i* of the left matrix with column *j* of the right one.

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

- Matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \vdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} & & & & & \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ & & & & & \end{bmatrix}$$

- **Notes:** (1) $A+B = B+A$, (2) $AB \neq BA$

Ex.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$
$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$
$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row *i* of **A** with column *j* of **B** – row by column multiplication

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$, Evaluate $C = AB$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1\times(-1) + 2\times 2 + 3\times 5 = 18 \\ c_{12} = 1\times 2 + 2\times 3 + 3\times 0 = 8 \\ c_{21} = 0\times(-1) + 1\times 2 + 4\times 5 = 22 \\ c_{22} = 0\times 2 + 1\times 3 + 4\times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 22 & 3 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \qquad AB = C = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}_{4\,x\,2}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 1 \cdot 2 + 0 \cdot 1 + 4 \cdot 3 = 14$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 1 \cdot 4 + 0 \cdot 1 + 4 \cdot 0 = 4$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 3 = 8$$

Properties of matrix addition and scalar multiplication:

If $A, B, C \in M_{m \times n}$, c, d : scalar, then

(1) $A+B = B + A$ Commutative law for addition

(2) $A + (B + C) = (A + B) + C$ Associative law for addition

(3) $(cd)A = c(dA)$ Associative law for scalar multiplication

(4) $c(A+B) = cA + cB$ Distributive law 1 for scalar multiplication

(5) $(c+d)A = cA + dA$ Distributive law 2 for scalar multiplication

- Properties of zero matrices:

If $A \in M_{m\times n}$, $c: scalar$

then (1) $A + 0_{m\times n} = A$

(2) $A + (-A) = 0_{m\times n}$

(3) $cA = 0_{m\times n} \Rightarrow c = 0 \ or \ A = 0_{m\times n}$

- Zero matrix: $0_{m\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$

- Notes:

(1) $0_{m\times n}$: the additive identity for the set of all $m\times n$ matrices

(2) $-A$: the additive inverse of A

Properties of Matrix Multiplication:

If A, B and C are two matrices and the matrix multiplications are defined

$$then: (1)\ (AB)C = A(BC)$$

$$(2)\ A(B+C) = AB + AC$$

$$(3)\ (A+B)C = AC + BC$$

$$(4)\ AI_n = A$$

$$(5)\ I_m A = A$$

Note:

- $AB = 0$ NOT necessarily imply $A = 0$ or $B = 0$
- $AB = AC$ NOT necessarily imply $B = C$

- Example

Show that *AB* and *BA* are not equal for the matrices.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

$$\Rightarrow \; AB \neq BA$$

- Real number:

$$ac = bc,\ c \neq 0$$

$$\Rightarrow\ a = b \qquad \text{(Cancellation law)}$$

Matrix:

$$AC = BC \qquad C \neq 0$$

(1) **If *C* is invertible**, then $A = B$

(2) If C is not invertible, then $A \neq B$ (Cancellation is not valid)

- Example: An example in which cancellation is not valid

Show that $AC=BC$

$$A=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix},\quad C=\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}$$

C is noninvertible, (*i.e.*, row 1 and row 2 are not independent)

Sol:

$$AC=\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

$$BC=\begin{bmatrix}2 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 2\end{bmatrix}=\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}$$

So $AC=BC$

But $A\neq B$

Powers of a Square Matrix A

$$A^r = \underbrace{AA\cdots A}_{\text{r times}}$$

Example

Let $B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, find the element that is in the third row and second column of the matrix $B^3 - 2I_3$