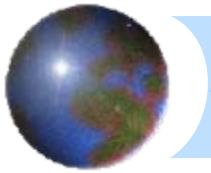
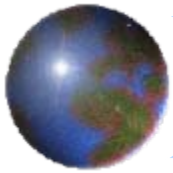


## **10.2 The Algebra of Matrices**



## Objectives

- Operations on Matrices: Sums, Scalar Products;
- Matrix Multiplication; Mechanics of Matrix Multiplication



# Operations on Matrices

- Equal matrices:

$$\text{Let } A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}$$

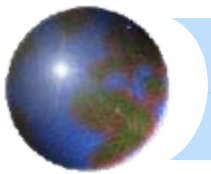
$$A = B \text{ if and only if } a_{ij} = b_{ij} \quad \forall 1 \leq i \leq m, 1 \leq j \leq n$$

- Ex

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

If  $A = B$

Then  $a = 1, b = 2, c = 3, d = 4$



**Definition** The *sum* of two same-sized matrices is their entry-by-entry sum. The *scalar multiple* of a matrix is the result of entry-by-entry scalar multiplication.

- **Matrix addition:**

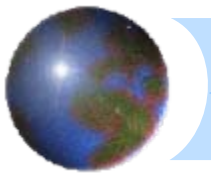
$$\text{If } A = [a_{ij}]_{m \times n}, \quad B = [b_{ij}]_{m \times n}$$

$$\text{Then } A + B = [a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$$

- **Scalar Multiplication:**

$$\text{If } A = [a_{ij}]_{m \times n}, \quad c : \text{scalar}$$

$$\text{Then } cA = [ca_{ij}]_{m \times n}$$



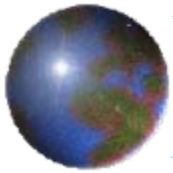
- Matrix subtraction:

$$A - B = A + (-1)B$$

- Example: Matrix addition

$$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1+1 & 2+3 \\ 0-1 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-1 \\ -3+3 \\ -2+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



# Matrix Multiplication

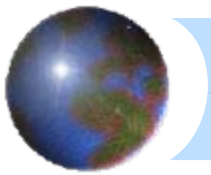
● Matrix multiplication of matrices  $A_{m \times k}$  and  $B_{k \times n}$

➤ number of columns of the first must equal number of rows of the second

the **product** is a matrix, denoted  $AB = C$

The  $i, j$  entry of the matrix product is the dot product of row  $i$  of the left matrix with column  $j$  of the right one.

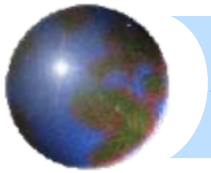
$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$



## ■ Matrix multiplication:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \vdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \end{bmatrix}$$

- **Notes:** (1)  $A+B = B+A$ , (2)  $AB \neq BA$



Ex.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$

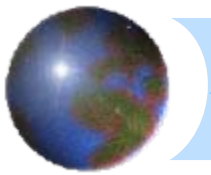
$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$

$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$

$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row  $i$  of  $\mathbf{A}$  with column  $j$  of  $\mathbf{B}$  – row by column multiplication

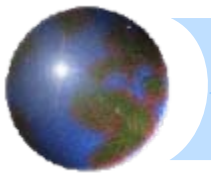




Example:  $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}$   $B = \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix}$  Evaluate  $C = AB$ .

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} \Rightarrow \begin{cases} c_{11} = 1 \times (-1) + 2 \times 2 + 3 \times 5 = 18 \\ c_{12} = 1 \times 2 + 2 \times 3 + 3 \times 0 = 8 \\ c_{21} = 0 \times (-1) + 1 \times 2 + 4 \times 5 = 22 \\ c_{22} = 0 \times 2 + 1 \times 3 + 4 \times 0 = 3 \end{cases}$$

$$C = AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 3 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 22 & 3 \end{bmatrix}$$



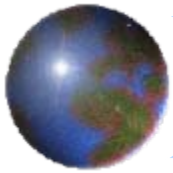
## Example

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix} \quad AB = C = \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \\ 8 & 2 \end{bmatrix}_{4 \times 2}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = 1 \cdot 2 + 0 \cdot 1 + 4 \cdot 3 = 14$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = 1 \cdot 4 + 0 \cdot 1 + 4 \cdot 0 = 4$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 3 = 8$$



## Properties of matrix addition and scalar multiplication:

If  $A, B, C \in M_{m \times n}$ ,  $c, d$  : scalar, then

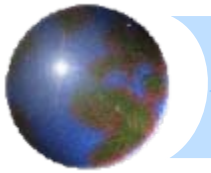
(1)  $A+B = B + A$  Commutative law for addition

(2)  $A + (B + C) = (A + B) + C$  Associative law for addition

(3)  $(cd)A = c(dA)$  Associative law for scalar multiplication

(4)  $c(A+B) = cA + cB$  Distributive law 1 for scalar multiplication

(5)  $(c+d)A = cA + dA$  Distributive law 2 for scalar multiplication



- Properties of zero matrices:

If  $A \in M_{m \times n}$ ,  $c : \text{scalar}$

then (1)  $A + 0_{m \times n} = A$

(2)  $A + (-A) = 0_{m \times n}$

(3)  $cA = 0_{m \times n} \Rightarrow c = 0 \text{ or } A = 0_{m \times n}$

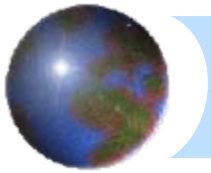
- Zero matrix:  $0_{m \times n}$

$$= \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix}$$

- Notes:

(1)  $0_{m \times n}$ : the additive identity for the set of all  $m \times n$  matrices

(2)  $-A$ : the additive inverse of  $A$



## ■ Properties of Matrix Multiplication:

If  $A$ ,  $B$  and  $C$  are two matrices and the matrix multiplications are defined

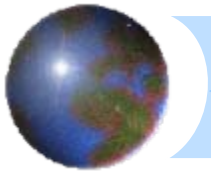
*then:* (1)  $(AB)C = A(BC)$

(2)  $A(B + C) = AB + AC$

(3)  $(A + B)C = AC + BC$

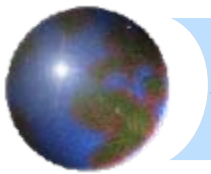
(4)  $AI_n = A$

(5)  $I_m A = A$



Note:

- $AB = 0$  NOT necessarily imply  $A = 0$  or  $B = 0$
- $AB = AC$  NOT necessarily imply  $B = C$



## ■ Example

Show that  $AB$  and  $BA$  are not equal for the matrices.

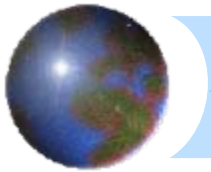
$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

Sol:

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}$$

$$\Rightarrow AB \neq BA$$



■ Real number:

$$ac = bc, c \neq 0$$

$$\Rightarrow a = b \quad (\text{Cancellation law})$$

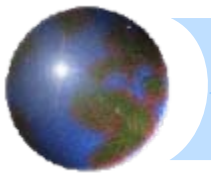
Matrix:

$$AC = BC \quad C \neq 0$$

(1) **If  $C$  is invertible**, then  $A = B$

(2) If  $C$  is not invertible, then  $A \neq B$  (Cancellation is not valid)





- Example: An example in which cancellation is not valid

Show that  $AC=BC$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}$$

**C is noninvertible,  
(i.e., row 1 and row 2  
are not independent)**

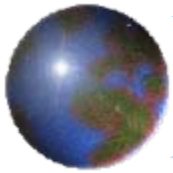
Sol:

$$AC = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$$

So  $AC = BC$

But  $A \neq B$



# Powers of a Square Matrix A

$$A^r = \underbrace{AA \cdots A}_{r \text{ times}}$$

## Example

Let  $B = \begin{bmatrix} 3 & -1 & 0 \\ 2 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ , find the element that is in the third row and second column of the matrix  $B^3 - 2I_3$