8.2 THE ELLIPSE

OBJECTIVES

1. Define an Ellipse
2. Find the Standard Form of the Equation of an Ellipse
3. Find the Major and Minor Axis
4. Find the Eccentricity of an Ellipse
5. Graph an Ellipse

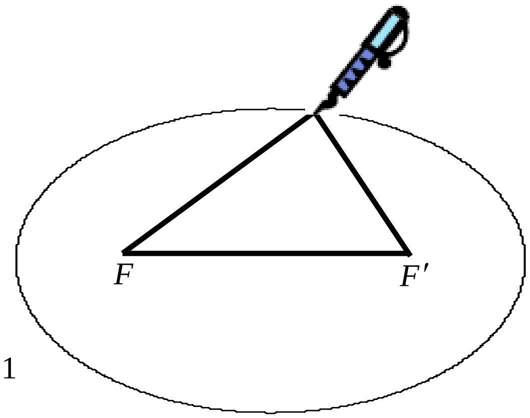


Figure 5.2.1

An ellipse can be constructed by placing two points at F and F' as shown in Figure 5.2.1 and placing a loop of a string over them. Pull the string taut with the point of a pencil, and then move the pencil keeping the string taut. The figure generated is an ellipse. From this construction, observe that the sum of the distances from the two fixed points to the point P is always the same, since the loop of a string is kept taut. This property characterizes the ellipse.

Definition of an Ellipse

An ellipse is the set of all points in a plane the sum of whose distances from two fixed points in the plane is constant.
Each of the fixed points is called a focus (plural: **foci**) of the ellipse. The midpoint of the line through the foci is called the **center** of the ellipse. We use the center of the ellipse to locate the ellipse in the plane, as we used the vertex to locate the parabola.

• Ellipses with Center at Origin (0, 0)

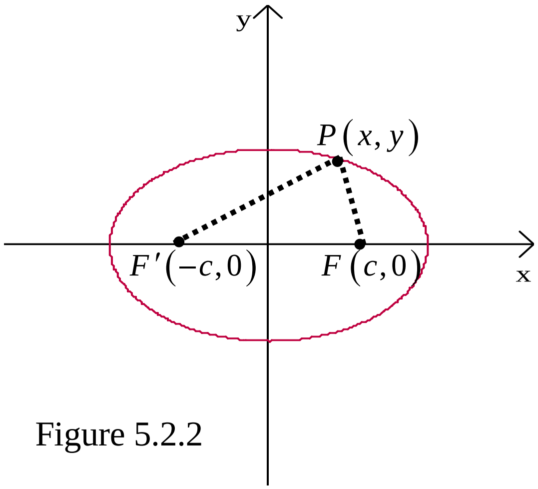


Figure 5.2.2

To obtain a simple equation for an ellipse, let us choose the foci to be at $F(c,0)$ and $F'(-c,0)$ where $c>0$, center at origin (0, 0), and the constant sum of the distances of a point $P(x,y)$ on the ellipse from F and F' to be $2a$ as shown in Figure 5.2.2.

Thus if we apply the distance formula, we get

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$$

From which we get $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2-c^2}=1$ **(see the book)**

The triangle PFF' of Figure 5.2.2 has one side of length $2c$. The sum of the lengths of the other sides is $2a$. Thus $2a>2c$

$$\Rightarrow a>c \Rightarrow a^2>c^2 \Rightarrow a^2-c^2>0$$

Since a^2-c^2 is positive, we may replace it by another positive number, b^2.Thus

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \text{ where } b^2=a^2-c^2$$

is the standard form of the equation of an ellipse centered at the origin with foci in the x-axis.

Properties of the Graph of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b^2=a^2-c^2$:(Horizontal Ellipse with Center (0,0)

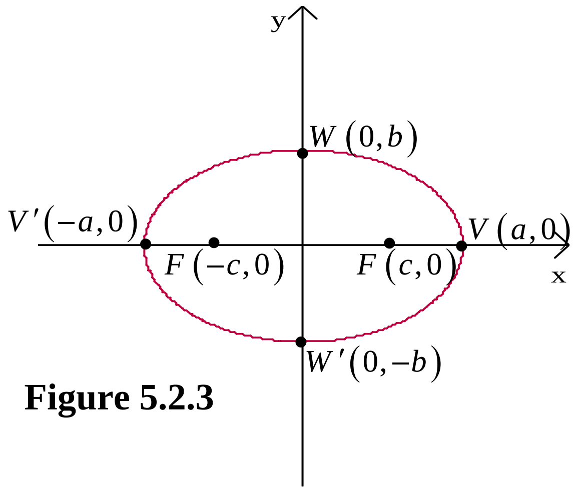


Figure 5.2.3

- Applying tests for symmetry, we see that the ellipse is symmetric with respect to the x-axis, the y-axis, and the origin**.**

- Letting $y=0$ in the standard form gives $x^2=a^2$ $\Rightarrow x=\pm a$.

$\Rightarrow$ the x-intercepts of the ellipse are $V(a,0)$ and $V'(-a,0)$ as shown in Figure 5.2.3.

- The points $V(a,0)$ and $V'(-a,0)$ are called the **vertices** of the ellipse.

- the line segment VV' is called the **major axis** whose length is $2a$.

- Similarly one can show that the y-intercepts of the ellipse are $W(0,b)$ and $W'(0,-b)$ as shown in Figure 5.2.3

- the line segment WW' is called the **minor axis** whose length is $2b$**.**

Properties of Vertical Ellipse with Center (0,0)

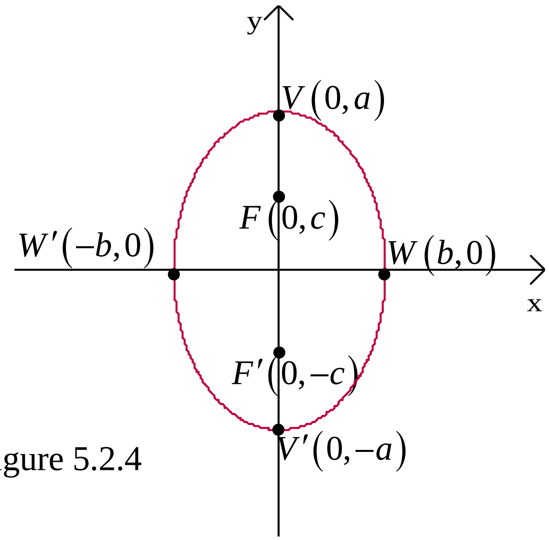


Figure 5.2.4

- vertices of the ellipse are $V(0,a)$ and $V'(0,-a)$
- endpoints of the minor axis are $W(b,0)$ and $W'(-b,0)$ as shown in Figure 5.2.4
- equation

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

Eccentricity

In addition to the quantities named, each ellipse is associated with a number, called the **eccentricity**. For any ellipse, the eccentricity is

$$e=\frac{c}{a}$$

The eccentricity of an ellipse satisfies the inequality

$$0<e<1$$

It gives a measure of the shape; the closer the eccentricity to 0, the more nearly circular is the ellipse as shown in Figure 5.2.5.

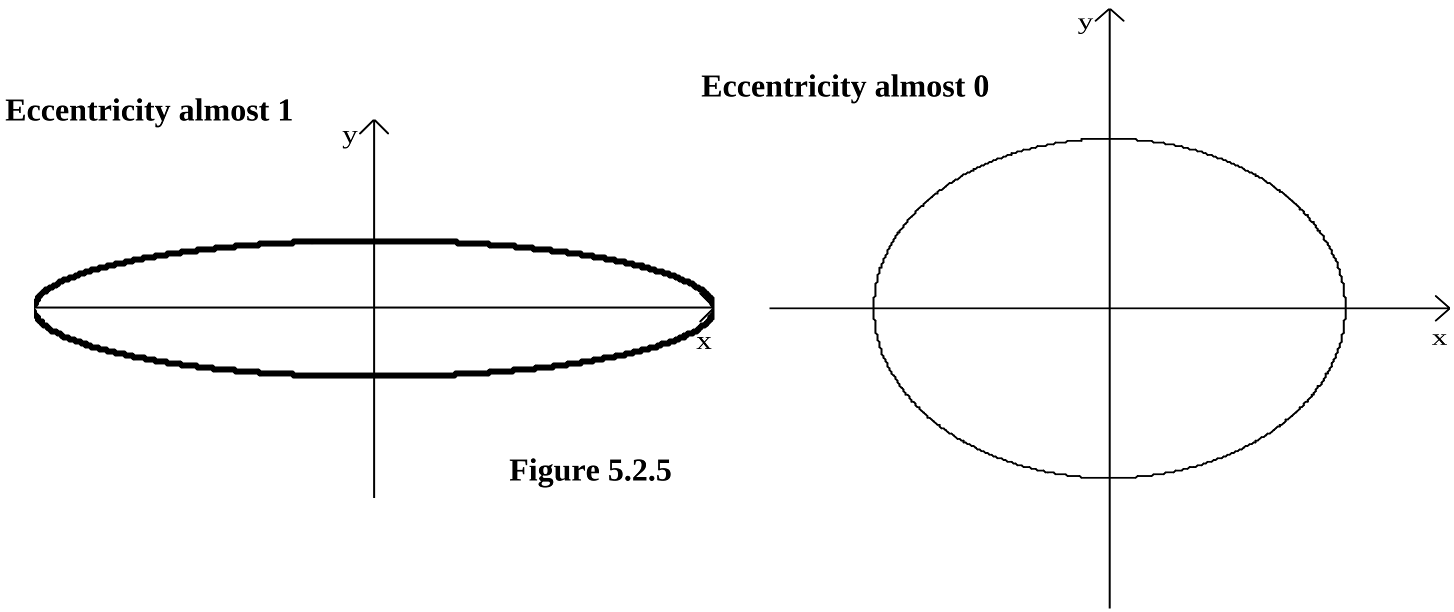


Figure 5.2.5

The preceding discussion may be summarized in the following table:

	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$
Center	$(0,0)$	$(0,0)$
Major axis	on the $x-$ axis Length $2a$	on the $y-$axis Length $2a$
Minor axis	on the y-axis Length $2b$	on the x-axis Length $2b$
Vertices	$(\pm a,0)$	$(0,\pm a)$
Endpoints of the minor axis	$(0,\pm b)$	$(\pm b,0)$
Foci	$(\pm c,0)$	$(0,\pm c)$
Eccentricity	$e=\frac{c}{a}$	$e=\frac{c}{a}$
In all of the above the number a is larger that b; a,b and c are related by $b^2=a^2-c^2$		

EXAMPLE 1

Sketch and discuss $9x^2 + 25y^2 = 225$.

Solution

First, we put the equation into the standard form by dividing through by 225:

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Now $a^2 = 25$, $b^2 = 9$, $c^2 = a^2 - b^2 = 16$. This ellipse has center $(0,0)$, vertices $(\pm 5,0)$, endpoints of minor axis $(0,\pm 3)$, major axis of length 6, and the eccentricity is $\frac{c}{b} = \frac{4}{5}$

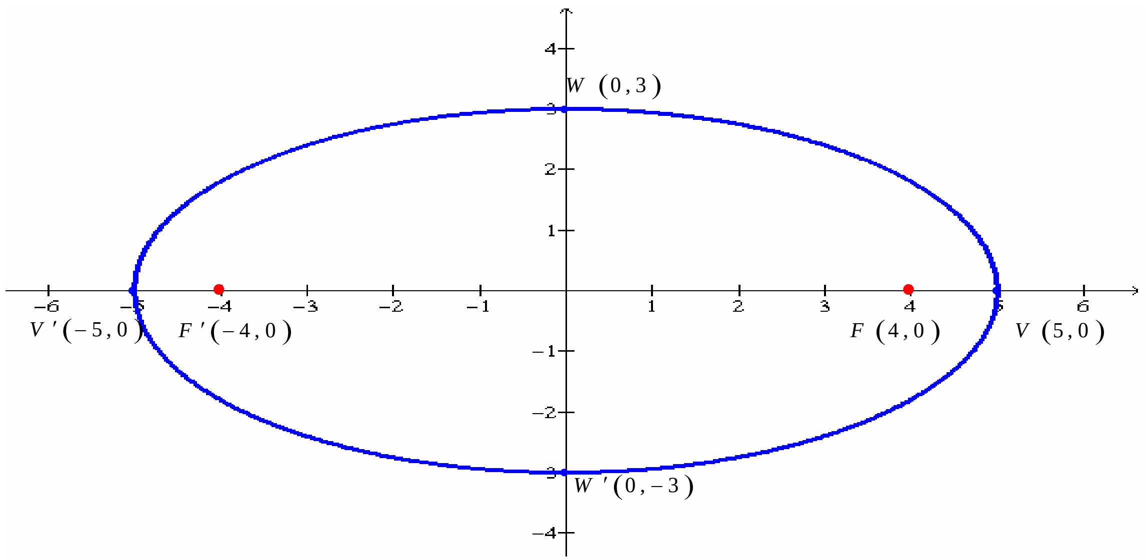


Figure 5.2.5

EXAMPLE 2

Find the equation in standard form of the ellipse that has its vertices at $(0, \pm 5)$ and eccentricity $\frac{3}{5}$.

Solution

From the given description, it is an ellipse with center at $(0,0)$, and the major axis and the foci are on the $y-$axis. With $a = 5$ and eccentricity $= \frac{c}{5} = \frac{3}{5} \Rightarrow c = 3 \Rightarrow$ foci at $(0, \pm 3)$ and $b^2 = a^2 - c^2 = 25 - 9 = 16$. Thus the required equation is $\frac{x^2}{16} + \frac{y^2}{25} = 1$ whose graph as shown in Figure 5.2.6

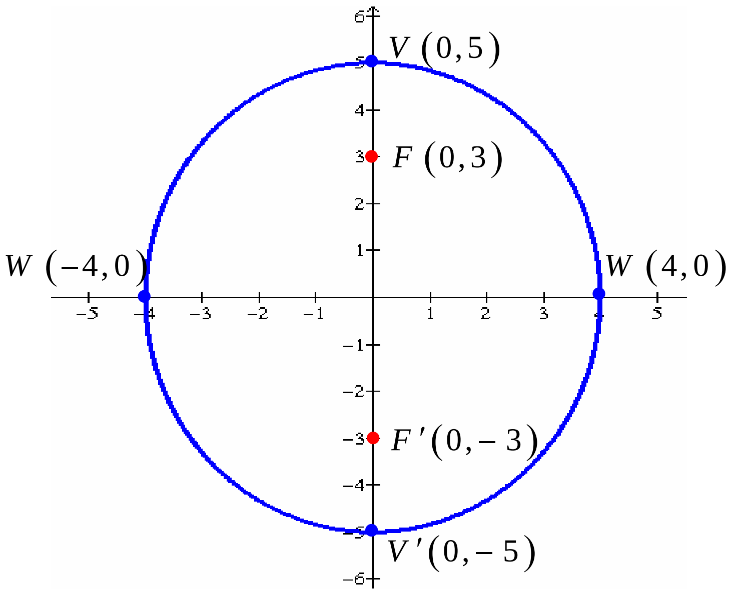


Figure 5.2.6

- **Ellipses with center at (h,k)**

If the center of the ellipse is at the point (h,k) and the **major and minor axes are parallel to the coordinate axes**, the $x-$axis and the $y-$axis, then there are two possible standard forms for the equation of the ellipse as illustrated in Figure 5.2.7 and Figure 5.2.8

Major axis parallel to the $x-$axis

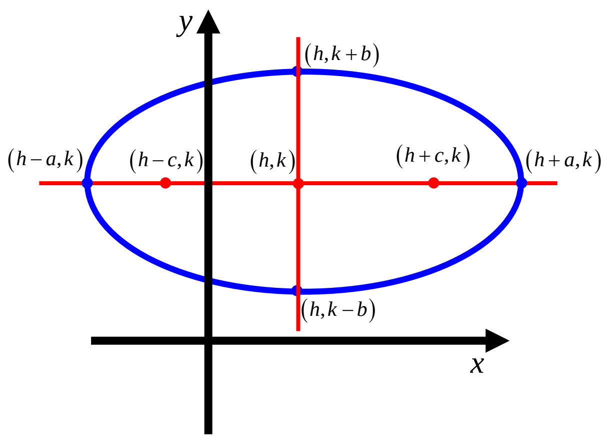
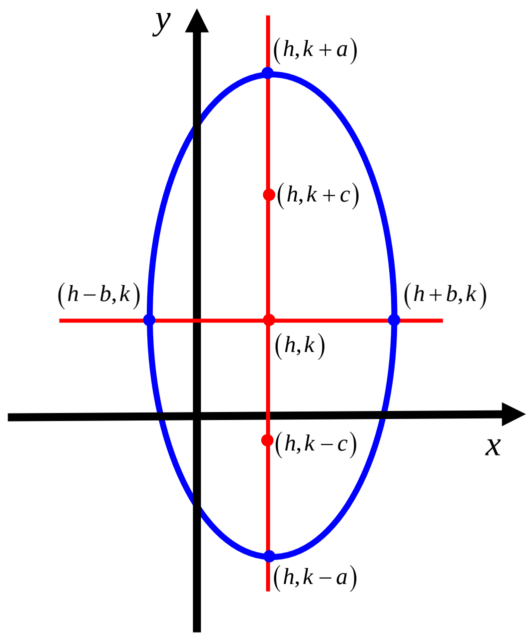


Figure 5.2.7

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,\ a>b$$

- Center at (h,k)
- Vertices at $(h\pm a,k)$
- Endpoints of minor axis at $(h,k\pm b)$
- Foci at $(h\pm c,k)$ where $c^2=a^2-b^2$
- Major axis is of length $2a$ and parallel to the $x-$axis where equation is $y=k$
- Minor axis is of length $2b$ and parallel to the $y-$axis where equation

	is $x = h$ • Eccentricity $\frac{c}{a}$

Major axis parallel to the $y-$axis	
 Figure 5.2.8	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1,\ a>b$ • Center at (h,k) • Vertices at $(h,k\pm a)$ • Endpoints of minor axis at $(h\pm b,k)$ • Foci at $(h,k\pm c)$ • Major axis is of length $2a$ and parallel to the $y-$axis where equation is $x=h$ • Minor axis is of length $2b$ and parallel to the $x-$axis where equation is $y=k$ • Eccentricity $\frac{c}{a}$

Common Properties:

- ❖ Center (h, k).
- ❖ Axes of symmetry are x=h and y=k.
- ❖ Distance from the center to each focus is c units.
- ❖ Distance between the two foci is 2c.

- ❖ Distance from the center to each vertex (endpoints of major axis) is a unit.
- ❖ Distance between the two vertices is 2a.

- ❖ Distance from the center to each endpoint of the minor axis is b units

EXAMPLE 3

Sketch and discuss the ellipse $\frac{(x-1)^2}{9}+\frac{(y+2)^2}{16}=1$.

Solution

The given equation represents an ellipse with vertical major axis, and whose center is at $(1,-2)$.

Now $a^2=16$ and $b^2=9$, so $a=4$ and $b=3$. Vertices at $(1,-2+4)=(1,2)$ and $(1,-2-4)=(1,-6)$.

Endpoints of the minor axis at $(1+3,-2)=(4,-2)$ and $(1-3,-2)=(-2,-2)$.

$c=\sqrt{16-9}=\sqrt{7}$, foci at $\left(1,-2+\sqrt{7}\right)$ and $\left(1,-2-\sqrt{7}\right)$.

Eccentricity $=\frac{\sqrt{7}}{4}$.

The graph of the ellipse as shown in Figure 5.2.9

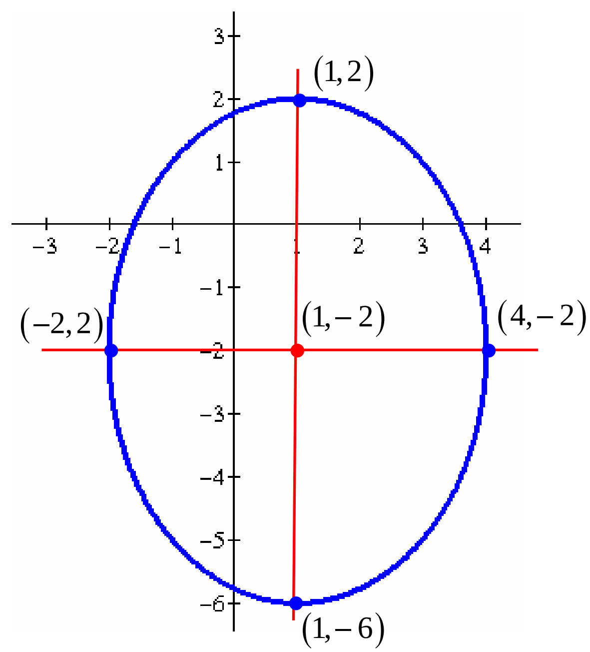


Figure 5.2.9

Squaring terms in the equation $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ or $\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$ and simplifying gives us an equation of the form $Ax^2+By^2+Cx+Dy+E=0$ where the coefficients are real numbers and both A and B are positive. Thus, if we start with such an equation, then by completing the squares we can put the equation in the standard form of an ellipse, as illustrated in the following example.

EXAMPLE 4

Show that the equation $9x^2+16y^2+18x-64y-71=0$ represents an ellipse, then sketch its graph.

Solution

The given equation can be written as

$$9\left(x^2+2x\right)+16\left(y^2-4y\right)-71=0$$

Now complete the squares:

$$9\left(x^2+2x+1\right)+16\left(y^2-4y+4\right)=71+9+64$$

$$\Rightarrow 9(x+1)^2+16(y-2)^2=144$$

$$\Rightarrow \frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$$

Which is the standard form of the equation of an ellipse with center at $(-1,2)$ and major axis on the horizontal line $y=2$ (since $16>9$).

Now $a^2 = 16$ and $b^2 = 16$, so $a = 4$, and $b = 3$ which gives us the ellipse in Figure 5.2.10

To find the foci:

$c^2 = a^2 - b^2 = 16 - 19 = 7 \Rightarrow c = \sqrt{7} \Rightarrow$ Foci at $\left(-1+\sqrt{7}, 2\right)$ and $\left(-1-\sqrt{7}, 2\right)$.

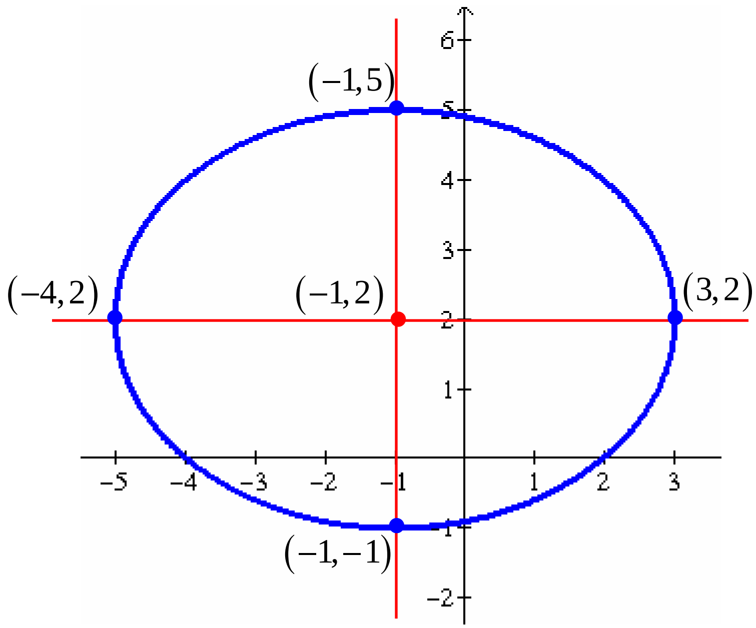


Figure 5.2.10