P.5 RATIONAL EXPRESSIONS

(التعابير الكسرية)

Objectives:

in this section, we learn about:

- Simplifying a Rational Expression
- Operations on Rational Expression
- Determining the Least Common Denominator(LCD)
- Simplifying Complex Fractions

Rational Expression (تعبير كسري): A fraction whose numerator and denominator are polynomials.

Domain of a Rational Expression (مجال تعريف التعبيرالكسري): the set of all real numbers *except* those for which the denominator is zero. i.e. $Domian = (-\infty,\infty) - \{x | \text{denominator} = 0\}$

Ex: Find the domain of the following expressions:

$a)\ \frac{x+2}{3-x}$

Sol.

$\text{the denominator } 3 - x = 0 \ if \ \ x = 3$

Thus the domain is $\{x | -\infty < x < \infty, x \neq 3\}$ or $(-\infty,\infty) - \{3\}$

or $(-\infty,3) \cup (3,\infty)$

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$b)\ \frac{x+2}{x^2 - 5x - 6}$

Sol.

The denominator $x^2 - 5x - 6 = 0 \ if \ (x-6)(x+1) = 0$

$\Rightarrow$ the zeros of the denominator are $x = -1, 6$

Thus the domain is $(-\infty,\infty) - \{-1,6\}$ or $(-\infty,-1) \cup (-1,6) \cup (6,\infty)$

or $\{x | -\infty < x < \infty, x \neq -1, 6\}$

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Properties of Rational Expressions

For all rational expressions where $Q \neq 0$ and $S \neq 0$, then:

Equality $\frac{P}{Q} = \frac{R}{S}$ *if and only if* $PS = RQ$

Sign $-\frac{P}{Q} = \frac{-P}{Q} = \frac{P}{-Q}$

Cancellation $\frac{P\cancel{R}}{Q\cancel{R}} = \frac{P}{Q}$.

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Simplify a Rational Expression:

- ❖Factor numerator and denominator completely
- ❖Use the cancellation property to remove common factors

A Rational Expression is simplified (أبسط صورة) (reduced form)
IF the numerator and denominator have no factors in common.

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Ex: Simplify $\dfrac{x^2-25}{x^2+7x+10}$

Sol:

$$\frac{x^2-25}{x^2+7x+10}=\frac{(x-5)\cancel{(x+5)}}{(x+2)\cancel{(x+5)}}$$

$$=\frac{(x-5)}{(x+2)}\ if\ x\neq -5$$

Simplifying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.

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Operations Rational Expressions

Operation	P, Q, R, and S are polynomials
Addition	$\dfrac{P}{R}+\dfrac{Q}{R}=\dfrac{P+Q}{R}$
Subtraction	$\dfrac{P}{R}-\dfrac{Q}{R}=\dfrac{P-Q}{R}$
Multiplication	$\dfrac{P}{R}\cdot\dfrac{Q}{S}=\dfrac{PQ}{RS}$
Division	$\dfrac{P}{R}\div\dfrac{Q}{S}=\dfrac{P}{R}\cdot\dfrac{S}{Q}=\dfrac{PS}{RQ}$

Notice the common denominator

Find the reciprocal and multiply

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Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerator and multiply the remaining factors in the denominator.

Note:

Arithmetic operations are defined on rational expressions just As they are on rational numbers.

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Ex: Simplify $\frac{x^2+2x+1}{x^3}\cdot\frac{6x^2-6x}{x^2-1}$

Sol:

$$=\frac{(x+1)(x+1)}{x^3}\cdot\frac{6x(x-1)}{(x-1)(x+1)}$$

$$=\frac{(x+1)\cancel{(x+1)}}{\cancel{x^3}}\cdot\frac{6\cancel{x}\cancel{(x-1)}}{\cancel{(x-1)}\cancel{(x+1)}}\qquad x\neq -1,0,1$$

$$=\frac{6(x+1)}{x^2}\qquad x\neq -1,0,1$$

Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerator and multiply the remaining factors in the denominator.

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Ex: Multiply and Simplify: $\frac{x-7}{x-1} \cdot \frac{x^2-1}{3x-21}$

$$\frac{x-7}{x-1} \cdot \frac{x^2-1}{3x-21}$$

Factor all numerators and denominators.

$$= \frac{x-7}{x-1} \cdot \frac{(x+1)(x-1)}{3(x-7)}$$

Because the denominator has factors of $x-1$ and $x-7$, $x \neq 1$ and $x \neq 7$.

$$= \frac{\cancel{(x-7)}}{\cancel{(x-1)}} \cdot \frac{(x+1)\cancel{(x-1)}}{3\cancel{(x-7)}}$$

Divide numerators and denominators by common factors.

$$= \frac{x+1}{3}, \ x \neq 1, \ x \neq 7$$

Multiply the remaining factors in the numerator and denominator.

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Ex: Divide and Simplify: $\frac{x^2-2x-8}{x^2-9} \div \frac{x-4}{x+3}$

$$\frac{x^2-2x-8}{x^2-9} \div \frac{x-4}{x+3}$$

$$= \frac{x^2-2x-8}{x^2-9} \cdot \frac{x+3}{x-4}$$

Multiply by the reciprocal

$$= \frac{(x-4)(x+2)}{(x+3)(x-3)} \cdot \frac{x+3}{x-4}$$

Factor numerator and denominator

$$= \frac{\overbrace{\cancel{(x-4)}}^{1}(x+2)}{\underbrace{\cancel{(x+3)}}_{1}(x-3)} \cdot \frac{\overbrace{\cancel{(x+3)}}^{1}}{\underbrace{\cancel{(x-4)}}_{1}}$$

$$= \frac{x+2}{x-3}, x \neq -3, x \neq 3, x \neq 4$$

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Least Common Denominator(LCD) (المقام المشترك الاصغر)

It is necessary to add rational expressions with **different** denominators

Finding the Least Common Denominator (LCD):

1. Factor each denominator completely.
2. Express repeated factors using powers
3. LCD = product of each factor raised to its largest power.

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Finding the Least Common Denominator (LCD):

1. Factor each denominator completely.
2. Express repeated factors using powers
3. LCD = product of each factor raised to its largest power.

Ex: Find the LCD $\frac{3x-4}{x^2+10x+25}$, $\frac{2x}{x^2-4x-5}$, $\frac{1}{1-x^2}$

$$\frac{3x-4}{(x+5)^2}\ ,\ \frac{2x}{(x-1)(x+5)},\ \frac{1}{(1-x)(x+1)}$$

Factor each denominator completely
Express repeated factors using powers

Can be made equal if minus sign is factored out

$$\frac{3x-4}{(x+5)^2}\ ,\ \frac{2x}{(x-1)(x+5)},\ \frac{1}{-(x-1)(x+1)}$$

$$\text{LCD} = -(x+5)^2(x-1)(x+1)$$

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Adding and Subtracting Expressions

1. Find the least common denominator.
2. Write all rational expressions in terms of the least common denominator.
3. Add or subtract the numerators, placing the resulting expression over the least common denominator.
4. If necessary, simplify the resulting expression.

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Ex: Simplify: $\frac{5}{9x^2}+\frac{1}{6x}-\frac{1}{x^2}$

Sol:

$$=\frac{5}{3^2x^2}+\frac{1}{3\cdot 2\cdot x}-\frac{1}{x^2} \qquad \text{LCD}=3^2\cdot 2\cdot x^2$$

$$=\frac{5}{3^2\cdot x^2}\frac{2}{2}+\frac{1}{3\cdot 2\cdot x}\frac{3\cdot x}{3\cdot x}-\frac{1}{x^2}\frac{3^2\cdot 2}{3^2\cdot 2}$$

$$=\frac{10+3x-18}{18x^2}$$

$$=\frac{3x-8}{18x^2}$$

Adding and Subtracting Expressions

1. Find the least common denominator.
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Ex: Simplify: $\dfrac{x+3}{x^2+x-2}+\dfrac{2}{x^2-1}$

Sol:

$$\frac{x+3}{x^2+x-2}+\frac{2}{x^2-1}$$

$$=\frac{x+3}{(x+2)(x-1)}+\frac{2}{(x+1)(x-1)}$$

LCD = $(x+2)(x-1)(x+1)$

$$=\frac{(x+3)}{(x+2)(x-1)}\cdot\frac{(x+1)}{(x+1)}+\frac{2}{(x-1)(x+1)}\cdot\frac{(x+2)}{(x+2)}$$

$$=\frac{x^2+4x+3+2x+4}{(x+2)(x-1)(x+1)}$$

$$=\frac{x^2+6x+7}{(x+2)(x-1)(x+1)},\ x\neq-2,\ x\neq1,\ x\neq-1$$

Adding and Subtracting Expressions

1. Find the least common denominator.
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Ex: Perform the following operation and simplify the result

$$\frac{1}{x}+\frac{2}{3x-1}\cdot\frac{3x^2+11x-4}{x-5}$$

Sol:

$$=\frac{1}{x}+\left(\frac{2}{3x-1}\cdot\frac{(3x-1)(x+4)}{x-5}\right)$$

Do the multiplication first

$$=\frac{1}{x}+\frac{2(x+4)}{x-5}$$

$$=\frac{1}{x}+\frac{2x+8}{x-5}\quad=\frac{1}{x}+\frac{2x+8}{x-5}\quad=\frac{x-5+x(2x+8)}{x(x-5)}$$

$$=\frac{2x^2+9x-5}{x(x-5)}=\frac{(2x-1)(x+5)}{x(x-5)}$$

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Complex Fractions (الكسور المركبة)

also called ***complex fractions***, have numerators or denominators containing one or more rational expressions.

Methods for Simplifying Complex Fractions

Method I

1. Determine the LCD of all denominators in the complex fraction.
2. Multiply both the numerator and the denominator of the complex fraction by the LCD.
3. If possible, simplify the resulting fraction.

Method II

1. Simplify the numerator to a single fraction and denominator to a single fraction.
2. Use the definition for dividing fractions, multiply the numerator by the reciprocal of the denominator.
3. If possible, simplify the resulting fraction.

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Ex: Simplify: $\dfrac{\frac{3}{x} - 2}{\frac{9}{x} - 4x}$

Sol:

$$= \frac{\left(\frac{3}{x} - 2\right)x}{\left(\frac{9}{x} - 4x\right)x}$$

$$= \frac{3-2x}{9-4x^2}$$

$$= \frac{3-2x}{(3-2x)(3+2x)}$$

$$= \frac{1}{3+2x}$$

Methods for Simplifying Complex Fractions

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Method II

1. Simplify the numerator to a single fraction and denominator to a single fraction.
2. Use the definition for dividing fractions, multiply the numerator by the reciprocal of the denominator.
3. If possible, simplify the resulting fraction.

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Ex: Simplify: $\dfrac{5-\dfrac{1}{x+2}}{1-\dfrac{3}{1+\dfrac{3}{x}}}$

Sol:

$$=\frac{\frac{5x+10-1}{x+2}}{1-\frac{3}{\frac{x+3}{x}}}=\frac{\frac{5x+9}{x+2}}{1-\frac{3x}{x+3}}$$

$$=\frac{\frac{5x+9}{x+2}}{\frac{-2x+3}{x+3}}=\frac{5x+9}{x+2}\,\frac{x+3}{-2x+3}$$

Methods for Simplifying Complex Fractions

Method II

1. Simplify the numerator to a single fraction and denominator to a single fraction.
2. Use the definition for dividing fractions, multiply the numerator by the reciprocal of the denominator.
3. If possible, simplify the resulting fraction.

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Ex: Simplify: $\dfrac{5-\dfrac{1}{x+2}}{1-\dfrac{3}{1+\dfrac{3}{x}}}$

Sol:

$$=\frac{5-\frac{1}{x+2}}{1-\frac{x}{x}\left(\frac{3}{1+\frac{3}{x}}\right)}=\frac{5-\frac{1}{x+2}}{1-\frac{3x}{x+3}}$$

$$=\frac{(x+2)(x+3)\left(5-\frac{1}{x+2}\right)}{(x+2)(x+3)\left(1-\frac{3x}{x+3}\right)}=\frac{(x+3)(5(x+2)-1)}{(x+2)((x+3)-3x)}=\frac{(x+3)(5x+9)}{(x+2)(-2x+3)}$$

Methods for Simplifying Complex Fractions

Method I

1. Determine the LCD of all denominators in the complex fraction.
2. Multiply both the numerator and the denominator of the complex fraction by the LCD.
3. If possible, simplify the resulting fraction.

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Ex: Simplify: $\dfrac{x^{-1}-y^{-1}}{x^{-3}-y^{-3}}$

Sol:

$$=\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^3}-\frac{1}{y^3}}=\frac{\frac{y-x}{xy}}{\frac{y^3-x^3}{x^3y^3}}$$

$$=\frac{y-x}{xy}\,\frac{x^3y^3}{y^3-x^3}$$

$$=\frac{y-x}{xy}\,\frac{x^3y^3}{(y-x)(y^2+yx+x^2)}$$

$$=\frac{x^2y^2}{(y^2+yx+x^2)}$$

Methods for Simplifying Complex Fractions

Method II

1. Simplify the numerator to a single fraction and denominator to a single fraction.
2. Use the definition for dividing fractions, multiply the numerator by the reciprocal of the denominator.
3. If possible, simplify the resulting fraction.

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Ex: Simplify: $\dfrac{x^{-1}-y^{-1}}{x^{-3}-y^{-3}}$

Sol:

$$=\frac{\frac{1}{x}-\frac{1}{y}}{\frac{1}{x^3}-\frac{1}{y^3}}=\frac{x^3y^3\left(\frac{1}{x}-\frac{1}{y}\right)}{x^3y^3\left(\frac{1}{x^3}-\frac{1}{y^3}\right)}$$

$$=\frac{\left(x^2y^3-x^3y^2\right)}{\left(y^3-x^3\right)}=\frac{x^2y^2\left(y-x\right)}{\left(y^3-x^3\right)}$$

$$=\frac{x^2y^2(y-x)}{(y-x)(y^2+yx+x^2)}$$

$$=\frac{x^2y^2}{(y^2+yx+x^2)}$$

Methods for Simplifying Complex Fractions

Method I

1. Determine the LCD of all denominators in the complex fraction.
2. Multiply both the numerator and the denominator of the complex fraction by the LCD.
3. If possible, simplify the resulting fraction.

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Ex: Simplify

Sol:

$$a)\ \left(-2^{-2}+3^{-1}\right)^{-1}$$

$$\left(-2^{-2}+3^{-1}\right)^{-1}=\left(-\frac{1}{4}+\frac{1}{3}\right)^{-1}$$

$$=\left(\frac{-3+4}{12}\right)^{-1}=\left(\frac{1}{12}\right)^{-1}\qquad =12$$

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$$b)\ \frac{a^2+9}{a^2-64}\div\frac{a^3-3a^2+9a-27}{a^2+5a-24}$$

Sol:

$$=\frac{a^2+9}{(a-8)(a+8)}\cdot\frac{a^2+5a-24}{a^3-3a^2+9a-27}$$

$$=\frac{a^2+9}{(a-8)(a+8)}\cdot\frac{(a+8)(a-3)}{\left(a^3-3a^2\right)+(9a-27)}$$

$$=\frac{a^2+9}{(a-8)(a+8)}\cdot\frac{(a+8)(a-3)}{a^2(a-3)+9(a-3)}$$

$$=\frac{\cancel{a^2+9}}{(a-8)\cancel{(a+8)}}\cdot\frac{\cancel{(a+8)}\cancel{(a-3)}}{\cancel{(a-3)}\cancel{\left(a^2+9\right)}}=\frac{1}{(a-8)}$$

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