P.4 FACTORING

(التحليل)

Objectives:

- ❖ Greatest Common Factor
- ❖ Factoring Trinomials
- ❖ Special Factoring
- ❖ Factoring by Grouping
- ❖ General Factoring

Factoring is the process of writing a polynomial as the ***product*** of two or more ***prime*** polynomials(***with lower degrees***)

Factoring over the set of integers, means that the coefficients in the factors are all integers.

Ex

$a)\ x^2-25\ \underline{\underline{rewritting\ as}}\ (x-5)(x+5)$ is an example over integers

$b)\ x^2-5\ \underline{\underline{rewritting\ as}}\ (x-\sqrt{5})(x+\sqrt{5})$ is an example over real nos.

Polynomials which *can not be factored* are called ***prime*** polynomials

(غير قابلة للتحليل)

The process of factoring a polynomial involves using one or more factoring techniques until the polynomial is factored completely.

Techniques of Factoring

1. Greatest Common Factors (العامل المشترك الاكبر)

In any factoring problem, the ***first step*** is to look for the *greatest common factor(GCF)*.

- The ***greatest common factor***, **(GCF)**,. is an expression that constitutes of the **product of *all common prime* factors raised to the smallest degree**.
  - This expression divides each term of the polynomial.(جميع الحدود تقبل القسمة عليه)
  - The distributive property in the reverse direction

$$ab + ac = a(b + c)$$

can be used to factor out the greatest common f acor.

Ex: Factor out the GCF.

$$6t^3 - 36t^2$$

Sol:

$$= 3 \cdot 2 \cdot t^3 - 3^2 \cdot 2^2 t^2$$

write each term as product of its prime factors

$$GCF = 3 \cdot 2 \cdot t^2$$

factor out the GCF = $6t^2$

$$= 6t^2 (t - 6)$$

Ex: Factor out the GCF.

Sol:

$$18x^3 + 27x^2 = 3^2 \cdot 2 \cdot x^3 + 3^3 \cdot x^2$$

$$= 9x^2 (2x + 3)$$

Ex: Factor out the GCF. $x^2(x + 3) + 5(x + 3)$

Sol:

$$= (x + 3)(x^2 + 5)$$

2. Factoring $ax^2 + bx + c$ (تحليل كثيرة حدود من الدرجة الثانية)

(Assume for the moment that there is no GCF)

1. Find two **First** terms whose product is ax^2.

$$(?\, x + \quad)(?\, x + \quad) = ax^2 + bx + c$$

2. Find two **Last** terms whose product is c.

$$(x + ?)\,(x + ?) = ax^2 + bx + c$$

3. By trial and error, perform steps 1 and 2 until the sum of the **Outside** product and **Inside** product is bx.

$$(?x + ?)\,(?x + ?) = ax^2 + bx + c$$

I

O

(sum of I + O)

If no such combinations exist, the polynomial is prime.

Ex: Factor

$a)\ x^2 + 4x + 3$

Sol:

Product is 3

$(\ x + \quad)(\ x + \quad)$

Product is 1

Since the constant term is positive, the factors have the same sign as the coefficient *b*

Try different factors of 1 and 3 so that the sum of the ***inner*** and ***outer product*** is *4x*

3x

$(\ x + 1\ )(\ x + 3\ )$

x

correct

Thus $x^2 + 4x + 3 = (x+1)(x+3)$

Ex: Factor

b) $6p^2 - 7p - 5$

Sol:

Product is -5

$(\; p - \quad)(\; p + \quad)$

Product is 6

one of the integers must be ***negative*** and one must be ***positive*** to give product -5

Attempt 1

2p

$(2p - 5)(3p + 1)$

-15p

incorrect

try different factors of 6 and -5 so that the sum of the ***inner*** and ***outer product*** is *-7p*

try 2 and 3 as factors of 6, and -5 and 1 as factors of -5

Attempt 2

3p

$(3p - 5)(2p + 1)$

-10p

correct

Thus $6p^2 - 7p - 5$ is factoerd as $(3p-5)(2p+1)$

$c)\ 4x^3 + 6x^2r - 10xr^2$

Sol: GCF is $2x$

$$= 2x\left(2x^2 + 3xr - 5r^2\right)$$

try to factor this trinomial

Product is $-5r^2$

$$(\ \ x - \ \ )(\ \ x + \ \ )$$

Product is 2

try different factors of 2 and $-5r^2$ so that the sum of the ***inner*** and ***outer product*** is $3xr$

5xr

$$(2x + 5r)(x - r)$$

-2xr

correct

Thus $4x^3 + 6x^2r - 10xr^2 = 2x(2x + 5r)(x - r)$

Factorization Theorem:

The trinomial ax^2+bx+c with integer coefficients *a*, *b* and *c* can be factored as the product of two binomial with integer coefficients

if and only if

$b^2 - 4ac$ is a perfect square.

i.e. $b^2 - 4ac = (\text{integer})^2$

Ex: Determine whether each trinomial is factorable over integers:

a) $8x^2 - 10x - 3$

Sol:

$$a = 8,\ b = -10,\ c = -3$$

$$b^2 - 4ac \quad = (-10)^2 - 4(8)(-3) \quad = 196 \quad = (14)^2$$

It is factorable over integers, using the methods we have seen, we find

$$8x^2 - 10x - 3 = (4x + 1)(2x - 3)$$

d) $2x^2 + 9x + 3$

Sol: $a = 2,\ b = 9,\ c = 3$

$$b^2 - 4ac \quad = (9)^2 - 4(2)(3) \quad = 57 \; \textit{which is not a perfect square}$$

The trinomial is not factorable over integers

3. Special Factoring (قوانين تحليل خاصة)

If a and b are **real numbers**, **variables**, or **algebraic expressions**, then:

Expression	can be factored as	
$a^2 - b^2$	$= (a + b)(a - b)$ [opposite sign]	Difference of Two Squares
$a^2 + 2ab + b^2$	$= (a + b)^2$	Perfect Square Trinomial
$a^2 - 2ab + b^2$	$= (a - b)^2$	Perfect Square Trinomial
$a^3 + b^3$	$= (a + b)(a^2 - ab + b^2)$ [Same sign; opposite sign]	Sum of Two Cubes
$a^3 - b^3$	$= (a - b)(a^2 + ab + b^2)$ [Same sign; opposite sign]	Difference of Two Cubes

Ex: Factor

$$m^2 - 9$$

Sol:

$$= m^2 - 3^2$$

$$= (m - 3)(m + 3)$$

No similar rule for a sum of squares

$$a^2 + b^2$$

Cannot be factored with real coefficients "prime"

Ex: Factor $x^6 - 4y^2$

Sol:

$$= (x^3)^2 - (2y)^2 = (x^3 + 2y)(x^3 - 2y)$$

Ex: Factor $25a^2 - 90ac + 81c^2$

Sol: $a = 25,\ b = -90,\ c = 81$

$$b^2 - 4ac = (90)^2 - 4 \cdot 25 \cdot 81 = 0\,(\textit{perfect square})$$

First term: $25a^2$ is a perfect square. $25a^2 = 5a \quad 5a$

Last term $81c^2$ is a perfect square. $81c^2 = (-9c) \quad (-9c)$

Middle term $2(5a)(-9c) = -90ac$

This is a perfect square trinomial.

$25a^2 - 90ac + 81c^2 = (5a - 9c)^2$

Ex: Factor $8x^4 + 27x$

Sol:

$$= x\left(8x^3 + 27\right)$$

$$= x\left((2x)^3 + 3^3\right)$$

$$= x(2x+3)\left(4x^2 - 6x + 9\right)$$

Note that this is a prime trinomial over integrs since

$$b^2 - 4ac < 0$$

4. **Factoring by Grouping** التحليل عن طريق التجميع

by a suitable rearrangement of terms, it may be possible to factor some polynomials that have four terms or more.

(this process is called ***factoring by grouping)***

Ex: Factor

Sol:

$$\underbrace{mx^2 + mx} - \underbrace{2x - 2}$$ Group

$$\left(mx^2 + mx\right) - \left(2x + 2\right)$$ Factor out the GCF

watch

Factor *mx*

Factor 2

$$mx(x+1) - 2(x+1)$$

$$(mx-2)(x+1)$$

Ex: Factor $x^3 + 4x^2 + 3x + 12$

Sol:

$= (x^3 + 4x^2) + (3x + 12)$ Group

$= x^2\,\mathbf{(x + 4)} + 3\,\mathbf{(x + 4)}$ Factor out the GCF

$= (x + 4)\,(x^2 + 3)$

Remember that this binomial is prime on integers?

5. General Factoring

Ex: Factor

$$a)\,4x^2+4x+1-y^2$$

Sol:

$$=\left(4x^2+4x+1\right)-y^2$$

$$=\left(2x+1\right)^2-y^2$$

$$=\left(\left(2x+1\right)+y\right)\left(\left(2x+1\right)-y\right)$$

$b)\ x^2+6xy+9y^2-x-3y$

Sol:

$$=\left(x^2+6xy+9y^2\right)+\left(-x-3y\right)$$

watch

$$=\left(x+3y\right)^2-\left(x+3y\right)$$

$$=\left(x+3y\right)\left(\left(x+3y\right)-1\right) \quad =\left(x+3y\right)\left(x+3y-1\right)$$

$c)\ 3x^4 + 4x^2 - 4$ quadratic in form (على شاكلة كثيرة الحدود من الدرجة الثانية)

Sol:

$a\square^2 + b\square + c$

since the given trinomial can be written as

$3\left(x^2\right)^2 + 4x^2 - 4$

let $u = x^2$ $\Rightarrow 3\left(x^2\right)^2 + 4x^2 - 4$ becomes

$3u^2 + 4u - 4$

now factor this trinomial using trial & error techniques

$= (3u - 2)(u + 2)$

$= (3u - 2)(u + 2)$, now sustitute back $u = x^2$

$= \left(3x^2 - 2\right)\left(x^2 + 2\right)$

Remember that these binomials are prime on integers?

$$d)\ 8 - x^6$$

Sol:

$$\Rightarrow 2^3 - \left(x^2\right)^3$$

Use difference of two squares

Remember that this binomial is prime on integers?

Remember that this trinomial is prime on integers?

$$= \left(2 - x^2\right)\left(4 + 2x^2 + x^4\right)$$

$e)\ x^{4n} - 2x^{2n} + 1$

Sol:

$$\Rightarrow \left(x^{2n}\right)^2 - 2x^{2n} + 1$$

Quadratic in form

let $u = x^{2n}$

$\Rightarrow x^{4n} - 2x^{2n} + 1$ becomes

$$u^2 - 2u + 1$$

Factor using trial & error technique

$$(u-1)(u-1)$$

, now substitute back $u = x^{2n}$

$$\left(x^{2n} - 1\right)\left(x^{2n} - 1\right)$$

$$\left(x^{2n}-1\right)\left(x^{2n}-1\right)$$

$$=\left(\left(x^{n}\right)^{2}-1^{2}\right)\left(\left(x^{n}\right)^{2}-1^{2}\right)$$

$$=\left(x^{n}-1\right)\left(x^{n}+1\right)\left(x^{n}-1\right)\left(x^{n}+1\right)$$

$$=\left(x^{n}-1\right)^{2}\left(x^{n}+1\right)^{2}$$

Ex: Find all positive values of k such that the following trinomial is a perfect square trinomial $36x^2 + kxy + 100y^2$

Sol:

This trinomial is a perfect square if $b^2 - 4ac = 0$

$$a = 36,\ b = k,\ c = 100$$

$$\Rightarrow (k)^2 - 4(36)(100) = 0$$

$$\Rightarrow \sqrt{k^2} = \sqrt{14400}$$

$$\Rightarrow |k| = 120$$

$$\Rightarrow k = \pm 120$$

$$\Rightarrow k = 120 \quad \textit{only}$$

Or you can rewrite the giver trinomial as

$$(6x)^2 + 2(6x)(10y) + (10y)^2$$

$$(6x)^2 + 120xy + (10y)^2$$

Thus $k=120$