

P.4 FACTORING

(التحليل)



Objectives:

- ❖ Greatest Common Factor
- ❖ Factoring Trinomials
- ❖ Special Factoring
- ❖ Factoring by Grouping
- ❖ General Factoring

Factoring is the process of writing a polynomial as the *product* of two or more *prime* polynomials(*with lower degrees*)

Factoring over the set of integers, means that the coefficients in the factors are all integers.

Ex a) $x^2 - 25$ rewriting as $(x-5)(x+5)$ is an example over integers

b) $x^2 - 5$ rewriting as $(x-\sqrt{5})(x+\sqrt{5})$ is an example over real nos.

Polynomials which *can not be factored* are called *prime* polynomials

(غير قابلة للتحويل)

The process of factoring a polynomial involves using one or more factoring techniques until the polynomial is factored completely.

Techniques of Factoring

1. Greatest Common Factors (العامل المشترك الاكبر)

In any factoring problem, the *first step* is to look for the *greatest common factor* (GCF).

■ The *greatest common factor*, (GCF), is an expression that constitutes of the **product of all common prime factors raised to the smallest degree.**

■ This expression divides each term of the polynomial. (جميع الحدود تقبل القسمة عليه)

■ The distributive property in the reverse direction

$$ab + ac = a(b + c)$$

can be used to factor out the greatest common factor.

Ex: Factor out the GCF.

$$6t^3 - 36t^2$$

Sol:

$$= 3 \cdot 2 \cdot t^3 - 3^2 \cdot 2^2 t^2$$

write each term as product of its prime factors

$$\text{GCF} = 3 \cdot 2 \cdot t^2$$

factor out the GCF = $6t^2$

$$= 6t^2 (t - 6)$$

Ex: Factor out the GCF.

Sol:

$$18x^3 + 27x^2 = 3^2 \cdot 2 \cdot x^3 + 3^3 \cdot x^2$$

$$= 9x^2 (2x + 3)$$

Ex: Factor out the GCF. $x^2(x + 3) + 5(x + 3)$

Sol:

$$= (x + 3)(x^2 + 5)$$

2. Factoring $ax^2 + bx + c$ (تحليل كثيرة حدود من الدرجة الثانية)

(Assume for the moment that there is no GCF)

1. Find two **First** terms whose product is ax^2 .

$$(\ ?x + \)(\ ?x + \) = ax^2 + bx + c$$

2. Find two **Last** terms whose product is c .

$$(x + ?)(x + ?) = ax^2 + bx + c$$

3. By trial and error, perform steps 1 and 2 until the sum of the **Outside** product and **Inside** product is bx .

$$(?x + ?)(?x + ?) = ax^2 + bx + c$$

(sum of I + O)

If no such combinations exist, the polynomial is prime.

Ex: Factor

a) $x^2 + 4x + 3$

Sol:

Product is 3

$$\left(\begin{array}{c} \downarrow \\ x + \end{array} \right) \left(\begin{array}{c} \downarrow \\ x + \end{array} \right)$$

Product is 1

3x

$$\left(\begin{array}{c} \downarrow \\ x + 1 \end{array} \right) \left(\begin{array}{c} \downarrow \\ x + 3 \end{array} \right)$$

x

Since the constant term is positive, the factors have the same sign as the coefficient b

Try different factors of 1 and 3 so that the sum of the *inner* and *outer product* is $4x$

correct

Thus $x^2 + 4x + 3 = (x + 1)(x + 3)$

Ex: Factor

$$b) 6p^2 - 7p - 5$$

Sol:

$$\begin{array}{c} \text{Product is } -5 \\ \downarrow \qquad \qquad \downarrow \\ (\quad p - \quad) (\quad p + \quad) \\ \uparrow \qquad \qquad \uparrow \\ \text{Product is } 6 \end{array}$$

one of the integers must be *negative* and one must be *positive* to give product -5

try different factors of 6 and -5 so that the sum of the *inner* and *outer product* is $-7p$

try 2 and 3 as factors of 6, and -5 and 1 as factors of -5

Attempt 1

$$\begin{array}{c} 2p \\ \downarrow \qquad \qquad \downarrow \\ (2p - 5) (3p + 1) \\ \uparrow \qquad \qquad \uparrow \\ -15p \quad \text{incorrect} \end{array}$$

Attempt 2

$$\begin{array}{c} 3p \\ \downarrow \qquad \qquad \downarrow \\ (3p - 5) (2p + 1) \\ \uparrow \qquad \qquad \uparrow \\ -10p \quad \text{correct} \end{array}$$

Thus $6p^2 - 7p - 5$ is factored as $(3p - 5)(2p + 1)$

$$c) 4x^3 + 6x^2r - 10xr^2$$

Sol: GCF is $2x$

$$= 2x(2x^2 + 3xr - 5r^2)$$

← try to factor this trinomial

Product is $-5r^2$

$$\left(\begin{array}{c} x - \\ \uparrow \end{array} \right) \left(\begin{array}{c} x + \\ \uparrow \end{array} \right)$$

Product is 2

try different factors of 2 and $-5r^2$ so that the sum of the *inner* and *outer product* is $3xr$

$5xr$

$$\left(\begin{array}{c} 2x + 5r \\ \uparrow \end{array} \right) \left(\begin{array}{c} x - r \\ \uparrow \end{array} \right)$$

$-2xr$

correct

Thus $4x^3 + 6x^2r - 10xr^2 = 2x(2x + 5r)(x - r)$



Factorization Theorem:

The trinomial $ax^2 + bx + c$ with integer coefficients a , b and c can be factored as the product of two binomial with integer coefficients

if and only if

$b^2 - 4ac$ is a perfect square.

i.e. $b^2 - 4ac = (\text{integer})^2$



Ex: Determine whether each trinomial is factorable over integers:

a) $8x^2 - 10x - 3$

Sol:

$$a = 8, b = -10, c = -3$$

$$b^2 - 4ac = (-10)^2 - 4(8)(-3) = 196 = (14)^2$$

It is factorable over integers, using the methods we have seen, we find

$$8x^2 - 10x - 3 = (4x + 1)(2x - 3)$$

d) $2x^2 + 9x + 3$

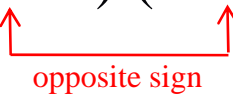
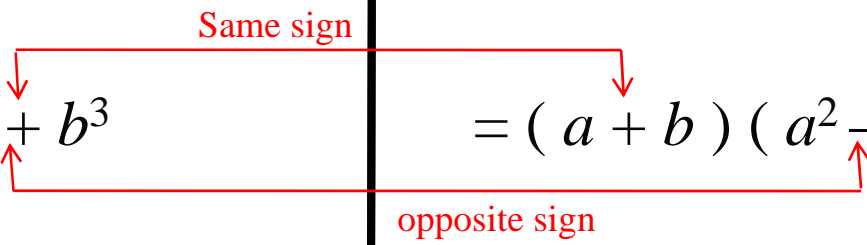
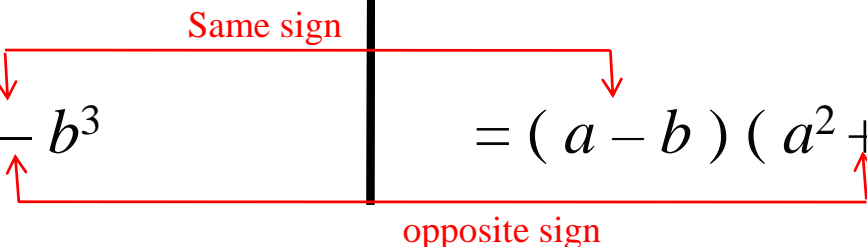
Sol: $a = 2, b = 9, c = 3$

$$b^2 - 4ac = (9)^2 - 4(2)(3) = 57 \text{ which is not a perfect square}$$

The trinomial is not factorable over integers

3. Special Factoring (قوانين تحليل خاصة)

If a and b are **real numbers**, **variables**, or **algebraic expressions**, then:

Expression	can be factored as	
$a^2 - b^2$	$= (a + b)(a - b)$ 	Difference of Two Squares
$a^2 + 2ab + b^2$	$= (a + b)^2$	Perfect Square Trinomial
$a^2 - 2ab + b^2$	$= (a - b)^2$	Perfect Square Trinomial
$a^3 + b^3$	$= (a + b)(a^2 - ab + b^2)$ 	Sum of Two Cubes
$a^3 - b^3$	$= (a - b)(a^2 + ab + b^2)$ 	Difference of Two Cubes



Ex: Factor

Sol:

$$\begin{aligned} & m^2 - 9 \\ &= m^2 - 3^2 \\ &= (m - 3)(m + 3) \end{aligned}$$

No similar rule for a sum of squares


$$a^2 + b^2$$

Cannot be factored with real coefficients “**prime**”

Ex: Factor $x^6 - 4y^2$

Sol:

$$= (x^3)^2 - (2y)^2 = (x^3 + 2y)(x^3 - 2y)$$



Ex: Factor $25a^2 - 90ac + 81c^2$

Sol: $a = 25, b = -90, c = 81$

$$b^2 - 4ac = (90)^2 - 4 \cdot 25 \cdot 81 = 0 \text{ (perfect square)}$$

First term: $25a^2$ is a perfect square. $25a^2 = 5a \quad 5a$

Last term $81c^2$ is a perfect square. $81c^2 = (-9c) \quad (-9c)$

Middle term $2(5a)(-9c) = -90ac$

This is a perfect square trinomial.

$$25a^2 - 90ac + 81c^2 = (5a - 9c)^2$$

Ex: Factor $8x^4 + 27x$

Sol:

$$= x(8x^3 + 27)$$

$$= x((2x)^3 + 3^3)$$

$$= x(2x + 3)(4x^2 - 6x + 9)$$

Note that this is a prime trinomial over integers since

$$b^2 - 4ac < 0$$

4. Factoring by Grouping التحليل عن طريق التجميع

by a suitable rearrangement of terms, it may be possible to factor some polynomials that have four terms or more.

(this process is called *factoring by grouping*)

Ex: Factor

Sol:

$$\underbrace{mx^2 + mx}_{\text{Factor } mx} - \underbrace{2x - 2}_{\text{Factor 2}}$$

Group

$$(mx^2 + mx) - (2x + 2) \quad \text{Factor out the GCF}$$


Factor mx

Factor 2

watch

$$mx(x + 1) - 2(x + 1)$$

$$(mx - 2)(x + 1)$$



Ex: Factor $x^3 + 4x^2 + 3x + 12$

Sol:

$$= (x^3 + 4x^2) + (3x + 12) \quad \text{Group}$$

$$= x^2(x + 4) + 3(x + 4) \quad \text{Factor out the GCF}$$

$$= (x + 4)(x^2 + 3)$$

↑
Remember that this
binomial is prime
on integers?



5. General Factoring

Ex: Factor


$$a) 4x^2 + 4x + 1 - y^2$$

Sol:

$$= (4x^2 + 4x + 1) - y^2$$

$$= (2x + 1)^2 - y^2$$

$$= ((2x + 1) + y)((2x + 1) - y)$$


$$b) x^2 + 6xy + 9y^2 - x - 3y$$

Sol:

$$= (x^2 + 6xy + 9y^2) + (-x - 3y) \quad \text{watch}$$

$$= (x + 3y)^2 - (x + 3y)$$

$$= (x + 3y)((x + 3y) - 1) = (x + 3y)(x + 3y - 1)$$

c) $3x^4 + 4x^2 - 4$ quadratic in form (على شاكلة كثيرة الحدود من الدرجة الثانية)

Sol:

$$a\boxed{}^2 + b\boxed{} + c$$

since the given trinomial can be written as

$$3(x^2)^2 + 4x^2 - 4$$

let $u = x^2 \Rightarrow 3(x^2)^2 + 4x^2 - 4$ becomes

$$3u^2 + 4u - 4$$

now factor this trinomial using trial & error techniques


$$= (3u - 2)(u + 2)$$

$$= (3u - 2)(u + 2)$$

, now substitute back $u = x^2$

$$= (3x^2 - 2)(x^2 + 2)$$

Remember that these binomials are prime on integers?



d) $8 - x^6$

Sol:


$$\Rightarrow 2^3 - (x^2)^3$$

Use difference of two squares

Remember that this
binomial is prime
on integers?

Remember that this
trinomial is prime
on integers?

$$= \left(2 - x^2\right) \left(4 + 2x^2 + x^4\right)$$



e) $x^{4n} - 2x^{2n} + 1$

Sol:

$$\Rightarrow (x^{2n})^2 - 2x^{2n} + 1$$

Quadratic in form

let $u = x^{2n}$

$$\Rightarrow x^{4n} - 2x^{2n} + 1 \text{ becomes}$$

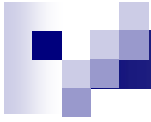
$$u^2 - 2u + 1$$

Factor using trial & error technique

$$(u - 1)(u - 1)$$

, now substitute back $u = x^{2n}$

$$(x^{2n} - 1)(x^{2n} - 1)$$



$$\begin{aligned} & (x^{2n} - 1)(x^{2n} - 1) \\ &= \left((x^n)^2 - 1^2 \right) \left((x^n)^2 - 1^2 \right) \\ &= (x^n - 1)(x^n + 1)(x^n - 1)(x^n + 1) \\ &= (x^n - 1)^2 (x^n + 1)^2 \end{aligned}$$

Ex: Find all positive values of k such that the following trinomial is a perfect square trinomial $36x^2 + kxy + 100y^2$

Sol:

This trinomial is a perfect square if $b^2 - 4ac = 0$

$$a = 36, b = k, c = 100$$

$$\Rightarrow (k)^2 - 4(36)(100) = 0$$

$$\Rightarrow \sqrt{k^2} = \sqrt{14400}$$

$$\Rightarrow |k| = 120$$

$$\Rightarrow k = \pm 120$$

$$\Rightarrow k = 120 \text{ only}$$

Or you can rewrite the given trinomial as

$$(6x)^2 + 2(6x)(10y) + (10y)^2$$

$$(6x)^2 + 120xy + (10y)^2$$

Thus $k=120$