P.2 INTEGER AND RATIONAL NUMBER EXPONENTS

(الأسس الصحيحة والنسبية)

Objectives:

In this lecture, you learn about:

1. Properties of Exponents
2. Scientific Notation
3. Rational Exponents and Radicals
4. Simplify Radical Expression

Definition of Natural Number Exponents

If a is a real number and n is a natural number, then

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}},$$

where a is the base (الأساس) and n is the exponent (الأس).

Ex:

a $4^3 = 4 \cdot 4 \cdot 4$

b. $-2^6 = -(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = -64$ (base is 2)

c. $(-2)^6 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 64$ (base is -2)

b^n : b to the power n *or* the n^{th} power of b

b^2 : b square

b^3 : b cube

Def. of b^0

For any nonzero real number b, $b^0 = 1, \quad b \neq 0$

Ex: a. $6^0 = 1$ $\quad$ $b.\ -6^0 = -(6^0) = -1$

$c.\ \left(x^2 + 5\right)^0 = 1$

Definition of b^{-n}

If $b \neq 0$ and n is a natural number, then $b^{-n} = \frac{1}{b^n}$ and $\frac{1}{b^{-n}} = b^n$

(This property applies to factors only(not to terms))

Ex.

$$a.\ \frac{y^{-2}x^{-3}}{2^{-1}t^{-5}} = \frac{2t^5}{y^2x^3}$$

$$b.\ \frac{y^{-2} + x^{-3}}{2^{-1}t^{-5}} = 2t^5\left(y^{-2} + x^{-3}\right)$$

Restriction Agreement

The expressions 0^0, 0^n (where n is a negative integer), and $\frac{a}{0}$ are all undefined.

Properties of Exponents

If *m, n, and p* are integers and *a*, and *b* are real numbers, then

Product $b^m b^n = b^{m+n}$

Quotient $\frac{a^m}{a^n} = a^{m-n}, if\ m > n\ or\ \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, if\ n > m$

Power $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}, \quad b \neq 0$

$$\left(b^m\right)^n = b^{mn}, \ \left(a^m b^n\right)^p = a^{mp} b^{np}$$

To simplify an expression involving exponents, write the expression in a from in which:

1. Each base occurs at most once
2. No powers of powers
3. No negative exponents occur

Ex: Simplify

$$a.\ \left(-3ab^4\right)\left(-6a^2b^{-4}\right)$$

Sol

$$=\left(-3\right)\left(-6\right)a^{1+2}b^{4+(-4)}$$

- Multiply the coefficients
- Multiply the variables by adding the exponents on the like bases

$$=18a^3b^0=18a^3$$

$$b.\ -2ab^2\left(-3a^2b\right)^3$$

Use the power property of exponents

Sol

$$=-2ab^2\left(-3\right)^3\left(a^2\right)^3b^3$$

$$=-2ab^2\left(-27\right)a^6b^3\quad=-2\left(-27\right)a^{1+6}b^{2+3}\quad=54a^7b^5$$

$$c.\left(\frac{-2x^2y}{6x^{-2}y^3}\right)^{-2}$$

Sol

$$=\left(\frac{-x^{2-(-2)}}{3y^{3-1}}\right)^{-2}$$

- Divide the coefficients
- Divide the variables by subtracting the exponents on the like bases

$$=\left(\frac{-x^4}{3y^2}\right)^{-2}$$

$$=\left(-\frac{3y^2}{x^4}\right)^{2}$$

Use the negative property of exponents

$$=\frac{9y^4}{x^8}$$

Use the power property of exponents

$$d.\left(\frac{(-a^2b^{-2})^{-1}c^3}{2a^2c^{-2}}\right)^{-2}$$

$$=\left(\frac{-a^{-2}b^2c^3}{2a^2c^{-2}}\right)^{-2} \quad =\left(\frac{-b^2c^5}{2a^4}\right)^{-2} \quad =\left(\frac{-2a^4}{b^2c^5}\right)^{2} \quad =\frac{4a^8}{b^4c^{10}}$$

Scientific Notation for Writing Numbers الصيغة العلمية

$$\pm a \times 10^{n}$$ ← integer

$$1 \le a < 10$$

Ex Write each of the following numbers in scientific notation

$$1)\ 3478.28 = 3.4828 \times 10^{3}$$

3 digits

$$2)\ 0.0000078 = 7.8 \times 10^{-6}$$

6digits

$$3)\ -345700000 = -3.457 \times 10^{8}$$

8 digits

Ex Write in decimal notation

1) $-3.253 \times 10^{7} = -32530000.$

7 decimal places

2) $4.25 \times 10^{-5} = 0.0000425$

5 decimal places

Ex Write the following expression in scientific notation

$$\frac{(3.2\times10^{-11})(2.7\times10^{18})}{1.2\times10^{-5}}$$

Sol

$$\frac{(3.2\times10^{-11})(2.7\times10^{18})}{1.2\times10^{-5}} = \frac{(3.2)(2.7)}{1.2}\times10^{-11+18-(-5)}$$

$$= 7.2\times10^{-11+18+5}$$

$$= 7.2\times10^{12}$$

Rational Exponents and Radicals الأسس النسبية والجذور

Definition of $b^{\frac{1}{n}}$

1. If n is an even positive integer and $b \geq 0$, then $b^{\frac{1}{n}}$ is the **nonnegative real** number such that $\left(b^{\frac{1}{n}}\right)^n = b$

2. If n is an odd positive integer, then $b^{\frac{1}{n}}$ is the **real number** such that $\left(b^{\frac{1}{n}}\right)^n = b$

Ex

a. $25^{1/2} = 5$ since $5^2 = 25$

b. $(-27)^{1/3} = -3$ since $(-3)^3 = -27$

c. $-4^{1/2} = -(4)^{\frac{1}{2}} = -2$

d. $(-4)^{1/2}$ is noe defined in real numbers e. $(-32)^{\frac{1}{5}} = -2$ since $(-2)^5 = -32$

Def. of $b^{\frac{m}{n}}$

For all positive integers *n*, all integers *m*, such that m/n is in **simplest form** and all real numbers *b* such that $b^{1/n}$ is a real number,

$$b^{m/n} = \left(b^{1/n}\right)^m = \left(b^m\right)^{1/n}$$

Ex: Evaluate $a)\ 125^{2/3}$

It can be evaluated in either of the following ways

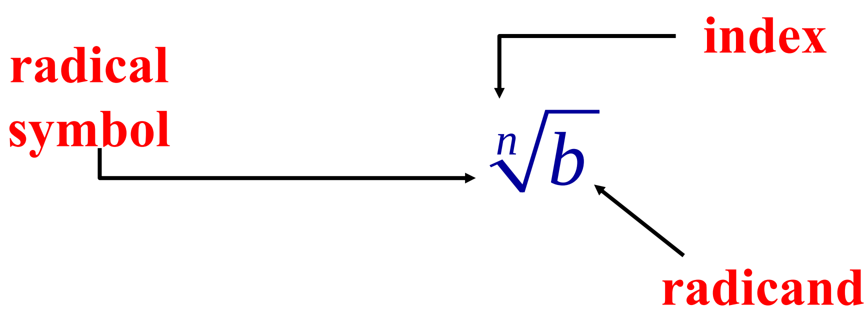
$$125^{2/3} = \left(125^2\right)^{1/3} = \left(15625\right)^{1/3} = 25$$

$$125^{2/3} = \left(125^{1/3}\right)^2 = (5)^2 = 25$$

$$b)\ 81^{-0.75} = \left(\frac{1}{81}\right)^{0.75} = \left(\frac{1}{81}\right)^{3/4} = \left(\left(\frac{1}{81}\right)^{1/4}\right)^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

Def. $\sqrt[n]{b}$

If n is a positive integer and b is areal number such that $b^{\frac{1}{n}}$ is a real number, then $\sqrt[n]{b} = b^{\frac{1}{n}}$



Common Roots

$\sqrt{b}$: the square root of b

$\sqrt[3]{b}$: the cube root of b

$\sqrt[n]{b}$: the n^{th} root of b, $for\ b > 3$

Def. of $\sqrt[n]{b^m}$

For all positive integers n, all integers m, and all real numbers b such that $\sqrt[n]{b}$ is a real number,

$$\sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m = b^{m/n}$$

Ex: Find the value of

$a)\ 125^{2/3} = \sqrt[3]{125^2} = 25$

alternatively, it can be evaluated as follows

$125^{2/3} = \left(\sqrt[3]{125}\right)^2 = (5)^2 = 25$

Easer to evaluate

$b)\ \left(\frac{4}{9}\right)^{-3/2} = \left(\sqrt{\frac{9}{4}}\right)^3 = \frac{27}{8}$

Def. of $\sqrt[n]{b^n}$

- If n is an even natural number and b is a real number, then

$$\sqrt[n]{b^n} = |b|$$

- If n is an odd natural number and b is a real number, then

$$\sqrt[n]{b^n} = b$$

Ex

$$\sqrt[4]{(-12)^4} = |-12| = 12$$

$$\sqrt[4]{16x^4} = \sqrt[4]{(2x)^4} = |2x|$$

$$\sqrt[3]{-x^3} = \sqrt[3]{(-x)^3} = -x$$

$$\sqrt[5]{(-12)^5} = -12$$

$$\sqrt[5]{32x^5} = \sqrt[5]{(2x)^5} = 2x$$

Ex: Simplify each exponential expression.

$$a.\ \left(\frac{32x^6y^3}{xy^{13}}\right)^{-0.4}$$

Sol

$$=\left(\frac{y^{10}}{32x^5}\right)^{2/5}=\left(\sqrt[5]{\frac{y^{10}}{32x^5}}\right)^2=\left(\frac{y^2}{2x}\right)^2=\frac{y^4}{4x^2}$$

$$b.\ \left(\frac{\sqrt[3]{x}\sqrt{y^{-4}z^3}}{xy^4z^{-9}}\right)^{-1/6},\ y>0, z>0$$

Sol

$$=\left(\frac{x^{1/3}y^{-2}z^3}{xy^4z^{-9}}\right)^{-1/6}=\left(\frac{z^{12}}{x^{2/3}y^6}\right)^{-1/6}=\left(\frac{x^{2/3}y^6}{z^{12}}\right)^{1/6}=\frac{x^{1/9}y}{z^2}$$

Properties of Radicals

If m and n are natural numbers and a and b are positive real numbers, then

Product	$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt[3]{8} \cdot \sqrt[3]{27} = \sqrt[3]{216} = 6$
Quotient	$\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}, \quad b \neq 0$	$\dfrac{\sqrt{8}}{\sqrt{2}} = \sqrt{\dfrac{8}{2}} = \sqrt{4} = 2$
Index	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt[4]{26}} = \sqrt[12]{26}$

An expression involving radicals is said to be in **simplest form** if it meets all of the following criteria.

1. The radicand contains only powers less than the index.

$$\sqrt{24} = \sqrt{4\cdot 6} = \sqrt{2^2\cdot 6} = 2\sqrt{6}$$

2. The index of the radical is as small as possible.

$$\sqrt[3]{\sqrt{125}} = \sqrt[6]{125} = \sqrt[6]{(5)^3} = 5^{3/6} = 5^{1/2} = \sqrt{5}$$

3. The denominators have been rationalized (no radicals appear in the denominator).

$$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

4. No fractions appear under the radicals sign.

$$\sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

Ex Simplify

a) $\sqrt{75}$

Sol

$$\sqrt{75} = \sqrt{5^2 \cdot 3} = 5\sqrt{3}$$

b) $\sqrt{24x^2y^3}$

Sol

$$\sqrt{24x^2y^3} = \sqrt{2^2x^2y^2 \cdot 6y} = 2|xy|\sqrt{6y}$$

Addition and Subtraction of Radicals

We add and subtract **like radicals** (the same radicand and the same index)

Ex10: Perform the following operations

$a)\ \sqrt{3}+5\sqrt{3}-3\sqrt{3}\quad =3\sqrt{3}$

$b)\ 3\sqrt[3]{xy^2}-2\sqrt[3]{xy^2}\quad =\sqrt[3]{xy^2}$

$c)\ 3\sqrt{75}-2\sqrt{48}\quad =3\sqrt{25\cdot 3}-2\sqrt{16\cdot 3}\quad =15\sqrt{3}-8\sqrt{3}=7\sqrt{3}$

Ex Simplify

$$a.\ 3x\sqrt[3]{8x^3y^4}+4y\sqrt[3]{64x^6y}$$

Solution

$$=3x\sqrt[3]{8x^3y^3\cdot y^1}+4y\sqrt[3]{4^3(x^2)^3\cdot y}$$

$$=3x\sqrt[3]{2^3x^3y^3}\cdot\sqrt[3]{y}+4y\sqrt[3]{4^3(x^2)^3}\sqrt{y}$$

$$=3x(2xy)\sqrt[3]{y}+4y(4x^2)\sqrt[3]{y}$$

$$=6x^2y\sqrt[3]{y}+16x^2y\sqrt[3]{y}$$

$$=22x^2y\sqrt[3]{y}$$

$$b.\ -3x\sqrt[3]{54x^4}+2\sqrt[3]{16x^7}$$

Sol

$$-3x\sqrt[3]{54x^4}+2\sqrt[3]{16x^7} = -3x\sqrt[3]{27x^3\cdot 2x}+2\sqrt[3]{8x^6\cdot 2x}$$

$$= -3x\sqrt[3]{27x^3}\cdot\sqrt[3]{2x}+2\sqrt[3]{8x^6}\cdot\sqrt[3]{2x}$$

$$= -3x\sqrt[3]{3^3\,x^3}\cdot\sqrt[3]{2x}+2\sqrt[3]{2^3\,x^6}\cdot\sqrt[3]{2x}$$

$$= -3x(3x)\sqrt[3]{2x}+2(2x^2)\sqrt[3]{2x}$$

$$= -9x^2\sqrt[3]{2x}+4x^2\sqrt[3]{2x} = -5x^2\sqrt[3]{2x}$$

$c.\ 2\sqrt[3]{40} - 3\sqrt[3]{135}$

Solution

$$2\sqrt[3]{8 \cdot 5} - 3\sqrt[3]{27 \cdot 5} = 2\sqrt[3]{8} \cdot \sqrt[3]{5} - 3\sqrt[3]{27} \cdot \sqrt[3]{5}$$

$$= 2\sqrt[3]{2^3} \cdot \sqrt[3]{5} - 3\sqrt[3]{3^3} \cdot \sqrt[3]{5}$$

$$= 2\sqrt[3]{2^3} \cdot \sqrt[3]{5} - 3\sqrt[3]{3^3} \cdot \sqrt[3]{5}$$

$$= 2(2)\sqrt[3]{5} - 3(3)\sqrt[3]{5}$$

$$= 4\sqrt[3]{5} - 9\sqrt[3]{5}$$

$$= -5\sqrt[3]{5}$$

Multiplication of Radicals

Multiplication of radicals is accomplished by using the distributive property.

Ex: Perform the following operations

$a.\ \sqrt{2}(\sqrt{8}+2\sqrt{2})$

$\sqrt{2}\sqrt{8}+\sqrt{2}\left(2\sqrt{2}\right) = \sqrt{16}+2(2)$ =4+4=8

$2)\ (\sqrt{3}+5)(\sqrt{3}-5)$

$=\sqrt{3}\sqrt{3}+\sqrt{3}(-5)+5\sqrt{3}+5(-5) = 3-5\sqrt{3}+5\sqrt{3}-25=-22$

$3)\ (3\sqrt{5y}-4)^2$

$3^2\left(\sqrt{5y}\right)^2+2(3\sqrt{5y})(-4)+(-4)^2 = 9(5y)-24\sqrt{5y}+16 = 45y-24\sqrt{5y}+16$

Rationalizing the Denominator إنطاق المقام

Rationalizing the denominator means writing the fraction with out radicals in the denominator.

Ex Rationalize the denominator.

a.) $\frac{1}{\sqrt{2}}$

Sol

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

conjugate

$$b)\ \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$$

Sol

$$=\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}\bullet\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}}=\frac{\left(\sqrt{2}-\sqrt{3}\right)^2}{\left(\sqrt{2}\right)^2-\left(\sqrt{3}\right)^2}$$

$$=\frac{2-2\sqrt{6}+3}{-1}=\frac{5-2\sqrt{6}}{-1}=-5+2\sqrt{6}$$

c) $\dfrac{1}{\sqrt[3]{x}}$

Sol

$$\frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}} \bullet \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{x^2}}{x}$$

c) $\dfrac{1}{\sqrt[5]{8x^2}}$

Sol

$$\frac{1}{\sqrt[5]{8x^2}} = \frac{1}{\sqrt[5]{2^3x^2}} \bullet \frac{\sqrt[5]{2^2x^3}}{\sqrt[5]{2^2x^3}} = \frac{\sqrt[5]{4x^3}}{\sqrt[5]{2^5x^5}} = \frac{\sqrt[5]{4x^3}}{2x}$$