4.1 INVERSE FUNCTIONS

Objectives:

In this section, we will learn about:

- Definition of Inverse Functions.
- Domain and Range of Inverse Functions.
- Finding an Equation of Inverse Functions.
- Graph of Inverse Functions.

Inverse Function

If the coordinates of the ordered pairs of a function g are the ***reverse*** of the coordinates of the ordered pairs of a function f, then g is said to be the *inverse function* of f.

Ex 1 Given the function $f = \{(1, -1), (2, 3), (-3, 1), (5, 2)\}$.

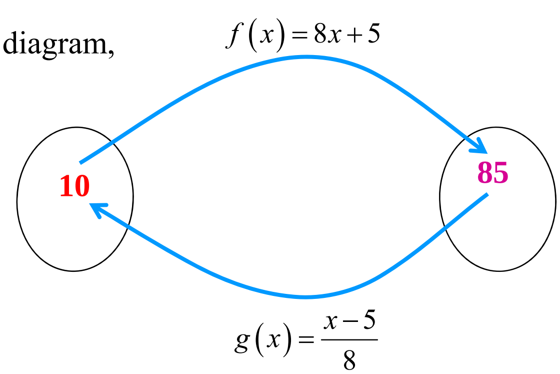
- The inverse relation of f is $g = \{(-1, 1), (3, 2), (1, -3), (2, 5)\}$.
- The domain of f is $\{1, 2, -3, 5\}$ and the range is $\{-1, 3, 1, 2\}$.
- The domain of g is $\{-1, 3, 1, 2\}$ and the range is $\{1, 2, -3, 5\}$.

Certain pairs of one-to-one functions **" undo" one another**. For example, if

$f(x) = 8x + 5$ and $g(x) = \dfrac{x-5}{8}$ then

$f(10) = 8 \cdot 10 + 5 = 85 \quad and \quad g(85) = \dfrac{85-5}{8} = 10$

See the diagram,



Notice that for this pair of functions, $f(x)=8x+5$ and $g(x)=\frac{x-5}{8}$

$f(g(2))=2$ and $g(f(2))=2$

In fact, for any value of *x,*

$f(g(x))=x$ and $g(f(x))=x$

or $(f \circ g)(x)=x$ and $(g \circ f)(x)=x$

Because of this property, g is called the ***inverse*** *of* f.

Composition of Inverse Functions Property:

If f is a ***one-to-one*** function, then g is the inverse function of f if and only if

$(f \circ g)(x)=f[g(x)]=x$ for all x in the domain of g

and

$(g \circ f)(x)=g[f(x)]=x$ for all x in the domain of f.

Condition for the Existence of the Inverse Function

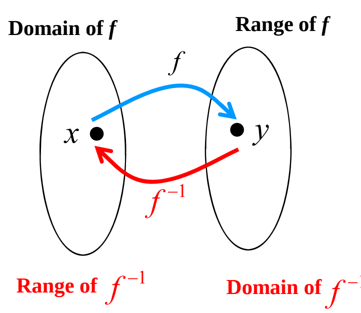
A function f has an **inverse function** if and only if f is a **one-to-one**.

Note:

A function that is either **increasing** or **decreasing** on its **entire** domain, must be one-to-one(hence, must has an inverse function)

Notes on the Inverse Function:

- The ***domain*** of the inverse function is the ***range*** of the original function.
- The range of the inverse function is the ***domain*** of the original function.
- If g is the inverse of f then f is also the inverse of g.
- The inverse function, of a function f, is denoted *by* f^{-1}
- $(f^{-1})^{-1} = f$



Caution: $f^{-1} \neq 1/f(x)$

Ex 2 Which of the following functions has an inverse function?

a) $f(x) = x^3 - 2$

b) $g(x) = x^2$

c) $h(x) = x^2, \quad x \geq 0$

Sol

In this example, we need to check whether a function is 1–1,

Reminder:
A function is 1-1 if there is at most one x-value for each y-value.

a) $f(x) = x^3 - 2$

$y = x^3 - 2$ **, solve for x** $\Rightarrow x = \sqrt[3]{y+2}$

Thus, the function is 1-1, and hence, has an inverse function.

b) $g(x) = x^2$

$y = x^2$ **, solve for x** $\Rightarrow x = \pm\sqrt{y}$?

Thus, the function is not 1-1, and hence, has no inverse function.

c) $h(x) = x^2, \quad x \geq 0$

$y = x^2$ **, solve for x** $\Rightarrow x = \sqrt{y}$?

Thus, the function is 1-1, and hence, has an inverse function.

Ex 3 Let $f(x)=x^3-1$ and $g(x)=\sqrt[3]{x+1}$. Is g the inverse function of f.

Sol

f is one-to-one function ? So it does have an inverse.

Now, check : $(f\,o\,g)(x)$ and $(g\,o\,f)(x)$

$$(f\,o\,g)(x)=f(g(x)) = \left(\sqrt[3]{x+1}\right)^3-1 = x$$

$$(g\,o\,f)(x)=g(f(x)) = \sqrt[3]{(x^3-1)+1} = x$$

Since $(f\,o\,g)(x)=x$ and $(g\,o\,f)(x)=x$

, the function g is the inverse of the function f.

Finding an Equation for f^{-1}, where f is One-to-One ?

Step 1 Write $y=f(x)$.

Step 2 Interchange x and y in the equation.

Step 3 Solve for y and replace y with f^{-1}.

Any restrictions on x or y should be considered.

Ex 4 Find the inverse functions of the following functions:

a) $f(x)=3x+5$.

Sol

$y=3x+5$ — Write $y=f(x)$

$x=3y+5$ — Interchange x and y

$y=\dfrac{x-5}{3}$ — Solve for y

$f^{-1}(x)=\dfrac{x-5}{3}$, $D_{f^{-1}}=(-\infty,\infty)$ — Write y as $f^{-1}(x)$

$b)\ f(x)=\dfrac{2x-1}{x+3}, x\neq -3$

Sol

$y=\dfrac{2x-1}{x+3}, x\neq -3$ Write $y=f(x)$

$x=\dfrac{2y-1}{y+3},\ y\neq -3$ Interchange x and y

$(y+3)x=2y-1$ Solve for y

$xy+3x=2y-1 \quad \Rightarrow xy-2y=-3x-1$

$\Rightarrow y(x-2)=-3x-1 \quad \Rightarrow y=\dfrac{-3x-1}{x-2}$

$\Rightarrow f^{-1}(x)=\dfrac{-3x-1}{x-2}$ Write y as $f^{-1}(x)$

$c)\ f(x)=x^2+4x+3,\quad x<-2$

Sol

$y=x^2+4x+3,\quad x<-2$ Write $y=f(x)$

$x=y^2+4y+3,\ y<-2$ Interchange x and y

$y^2+4y=x-3$ Solve for y

$y^2+4y+4=x-3+4$

$(y+2)^2=x+1 \quad \Rightarrow |y+2|=\sqrt{x+1}$

$\Rightarrow y+2=-\sqrt{x+1}$ **Why negative?** $\Rightarrow y=-2-\sqrt{x+1}$

$f^{-1}(x)=-2-\sqrt{x+1}\quad ,D_{f^{-1}}=(-1,\infty)$ Write y as $f^{-1}(x)$

$d)\ g(x)=\sqrt{4-x},\quad x\le 4$

Sol

$y=\sqrt{4-x},\quad x\le 4$ — Write $y=f(x)$

$x=\sqrt{4-y},\quad y\le 4$ — Interchange x and y

$x^2=4-y,\quad y\le 4$ — Solve for y

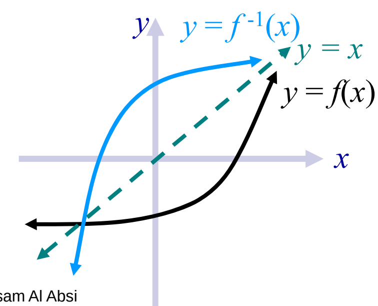
$y=4-x^2,\quad y\le 4$

$f^{-1}(x)=4-x^2\quad ,D_{f^{-1}}=[0,\infty)$ — Write y as $f^{-1}(x)$

Graphs of Inverse Functions

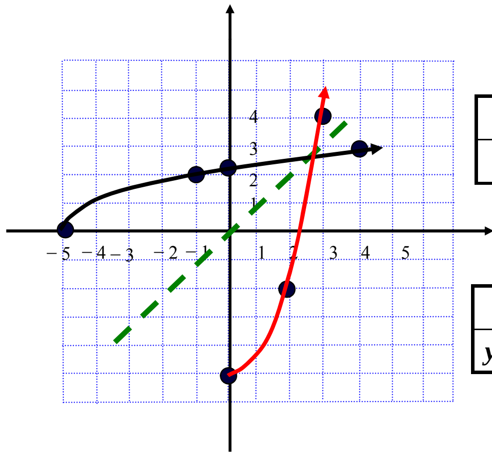
To sketch the graph of f^{-1}, we can use the following facts about inverses:

- The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a)
- The graph of f^{-1} is symmetric to the graph of f with respect to the line $y = x$.



Ex 5 Sketch the graph of f^{-1} using the graph $f(x)=\sqrt{x+5}$

Sol



Select some points on f

x	-5	-1	0	4
$y=f(x)$	0	2	2.5	3

points on f^{-1}

x	0	2	2.5	3
$y=f^{-1}(x)$	-5	-1	0	4

Ex 6 Let $f(x)=-x^2-3x+k$ such that $f^{-1}(x)$ exists. If $f^{-1}(2)=3$, then find k.

Sol

If $f^{-1}(2)=3$, then $f(3)=2$

$2=-(3)^2-3(3)+k$ Thus, $k=20$

Remember:

$$\boxed{y=f(x) \Leftrightarrow x=f^{-1}(y)}$$

Ex 7 If $f(x)=\dfrac{2x+1}{x-1}$, then find $(f\circ f^{-1})(5)+f^{-1}(1)$

Sol

$(f\circ f^{-1})(5)+f^{-1}(1)$

$=5$

$\dfrac{2x+1}{x-1}=1 \Rightarrow x=-2=f^{-1}(1)$

Thus, $(f\circ f^{-1})(5)+f^{-1}(1)=5-2=3$

Is 5 in $D_{f^{-1}}$?
equivalently,
Is 5 in R_f ?
$5=\dfrac{2x+1}{x-1}$
$x=2$
So, yes
$5\in R_f=D_{f^{-1}}$

Ex 8 If $f^{-1}(x) = 2 + \sqrt{x-1}, x \geq 1$ then find $f(4)$

Remember:

$$\boxed{\begin{array}{c} y = f(x) \\ \Leftrightarrow \\ x = f^{-1}(y) \end{array}}$$

Sol

Let $f(4) = y$, then $f^{-1}(y) = 4$

$\Rightarrow 4 = 2 + \sqrt{y-1}, y \geq 1$ Solve this radical equation

$\Rightarrow 2 = \sqrt{y-1}, y \geq 1$

$\Rightarrow 4 = y - 1, y \geq 1$ Solve this radical equation

$\Rightarrow y = 5$

$\Rightarrow f(4) = 5$