

4.1 INVERSE FUNCTIONS

Objectives:

In this section, we will learn about:

- Definition of Inverse Functions.
- Domain and Range of Inverse Functions.
- Finding an Equation of Inverse Functions.
- Graph of Inverse Functions.

Inverse Function

If the coordinates of the ordered pairs of a function g are the *reverse* of the coordinates of the ordered pairs of a function f , then g is said to be the *inverse function* of f .

Ex 1 Given the function $f = \{(1, -1), (2, 3), (-3, 1), (5, 2)\}$.

- The inverse relation of f is $g = \{(-1, 1), (3, 2), (1, -3), (2, 5)\}$.
- The domain of f is $\{1, 2, -3, 5\}$ and the range is $\{-1, 3, 1, 2\}$.
- The domain of g is $\{-1, 3, 1, 2\}$ and the range is $\{1, 2, -3, 5\}$.

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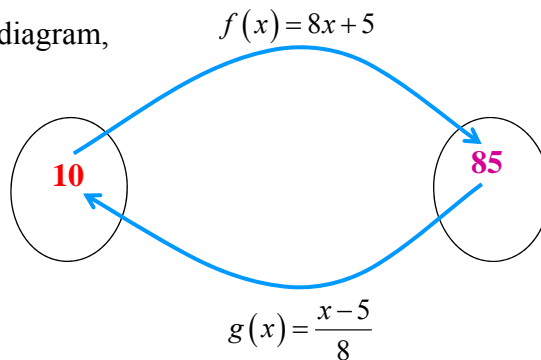
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Certain pairs of one-to-one functions “undo” one another. For example, if

$$f(x) = 8x + 5 \quad \text{and} \quad g(x) = \frac{x-5}{8} \quad \text{then}$$
$$f(10) = 8 \cdot 10 + 5 = 85 \quad \text{and} \quad g(85) = \frac{85-5}{8} = 10$$

See the diagram,



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Notice that for this pair of functions, $f(x) = 8x + 5$ and $g(x) = \frac{x-5}{8}$

$$f(g(2)) = 2 \quad \text{and} \quad g(f(2)) = 2$$

In fact, for any value of x ,

$$f(g(x)) = x \quad \text{and} \quad g(f(x)) = x$$

or

$$(f \circ g)(x) = x \quad \text{and} \quad (g \circ f)(x) = x$$

Because of this property, g is called the *inverse of f* .

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Composition of Inverse Functions Property:

If f is a *one-to-one* function, then g is the inverse function of f if and only if

$$(f \circ g)(x) = f[g(x)] = x \quad \text{for all } x \text{ in the domain of } g$$

and

$$(g \circ f)(x) = g[f(x)] = x \quad \text{for all } x \text{ in the domain of } f.$$

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Condition for the Existence of the Inverse Function

A function f has an **inverse function** if and only if f is a **one-to-one**.

Note:

A function that is either **increasing** or **decreasing** on its **entire** domain, must be one-to-one (hence, must have an inverse function)

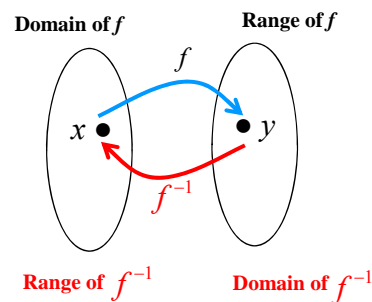
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Notes on the Inverse Function:

- The **domain** of the inverse function is the **range** of the original function.
- The range of the inverse function is the **domain** of the original function.
- If g is the inverse of f then f is also the inverse of g .
- The inverse function, of a function f , is denoted by f^{-1}
- $(f^{-1})^{-1} = f$



Caution: $f^{-1} \neq 1/f(x)$

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Ex 2 Which of the following functions has an inverse function?

a) $f(x) = x^3 - 2$

b) $g(x) = x^2$

c) $h(x) = x^2, x \geq 0$

Sol

In this example, we need to check whether a function is 1-1,

a) $f(x) = x^3 - 2$

$y = x^3 - 2$, solve for $x \Rightarrow x = \sqrt[3]{y + 2}$

Thus, the function is 1-1, and hence, has an inverse function.

Reminder:

A function is 1-1 if there is at most one x -value for each y -value.

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b) $g(x) = x^2$

$y = x^2$, solve for $x \Rightarrow x = \pm\sqrt{y}$?

Thus, the function is not 1-1, and hence, has no inverse function.

c) $h(x) = x^2, x \geq 0$

$y = x^2$, solve for $x \Rightarrow x = \sqrt{y}$?

Thus, the function is 1-1, and hence, has an inverse function.

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Ex 3 Let $f(x) = x^3 - 1$ and $g(x) = \sqrt[3]{x+1}$. Is g the inverse function of f .

Sol

f is one-to-one function? So it does have an inverse.

Now, check : $(f \circ g)(x)$ and $(g \circ f)(x)$

$$\begin{array}{l|l} (f \circ g)(x) = f(g(x)) & (g \circ f)(x) = g(f(x)) \\ = (\sqrt[3]{x+1})^3 - 1 & = \sqrt[3]{(x^3 - 1) + 1} \\ = x & = x \end{array}$$

Since $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$, the function g is the inverse of the function f .

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Finding an Equation for f^{-1} , where f is One-to-One ?

Step 1 Write $y = f(x)$.

Step 2 Interchange x and y in the equation.

Step 3 Solve for y and replace y with f^{-1} .

Any restrictions on x or y should be considered.

Ex 4 Find the inverse functions of the following functions:

a) $f(x) = 3x + 5$.

Sol

$$y = 3x + 5$$

Write $y = f(x)$

$$x = 3y + 5$$

Interchange x and y

$$y = \frac{x-5}{3}$$

Solve for y

$$f^{-1}(x) = \frac{x-5}{3}, \quad D_{f^{-1}} = (-\infty, \infty)$$

Write y as $f^{-1}(x)$

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$$b) f(x) = \frac{2x-1}{x+3}, x \neq -3$$

Sol

$$y = \frac{2x-1}{x+3}, x \neq -3$$

Write $y = f(x)$

$$x = \frac{2y-1}{y+3}, y \neq -3$$

Interchange x and y

$$(y+3)x = 2y-1$$

Solve for y

$$xy + 3x = 2y - 1 \quad \Rightarrow \quad xy - 2y = -3x - 1$$

$$\Rightarrow y(x-2) = -3x-1 \quad \Rightarrow \quad y = \frac{-3x-1}{x-2}$$

$$\Rightarrow f^{-1}(x) = \frac{-3x-1}{x-2}$$

Write y as $f^{-1}(x)$

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$$c) f(x) = x^2 + 4x + 3, x < -2$$

Sol

$$y = x^2 + 4x + 3, x < -2$$

Write $y = f(x)$

$$x = y^2 + 4y + 3, y < -2$$

Interchange x and y

$$y^2 + 4y = x - 3$$

Solve for y

$$y^2 + 4y + 4 = x - 3 + 4$$

$$(y+2)^2 = x+1 \quad \Rightarrow \quad |y+2| = \sqrt{x+1}$$

$$\Rightarrow y+2 = -\sqrt{x+1} \quad \text{Why negative?} \quad \Rightarrow \quad y = -2 - \sqrt{x+1}$$

$$f^{-1}(x) = -2 - \sqrt{x+1}, D_{f^{-1}} = (-1, \infty) \quad \text{Write } y \text{ as } f^{-1}(x)$$

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$$d) g(x) = \sqrt{4-x}, \quad x \leq 4$$

Sol

$$y = \sqrt{4-x}, \quad x \leq 4$$

Write $y = f(x)$

$$x = \sqrt{4-y}, \quad y \leq 4$$

Interchange x and y

$$x^2 = 4-y, \quad y \leq 4$$

Solve for y

$$y = 4-x^2, \quad y \leq 4$$

$$f^{-1}(x) = 4-x^2, \quad D_{f^{-1}} = [0, \infty)$$

Write y as $f^{-1}(x)$

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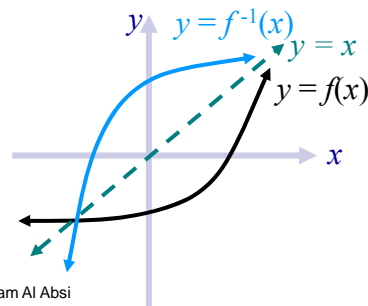
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Graphs of Inverse Functions

To sketch the graph of f^{-1} , we can use the following facts about inverses:

- The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a)
- The graph of f^{-1} is symmetric to the graph of f with respect to the line $y = x$.



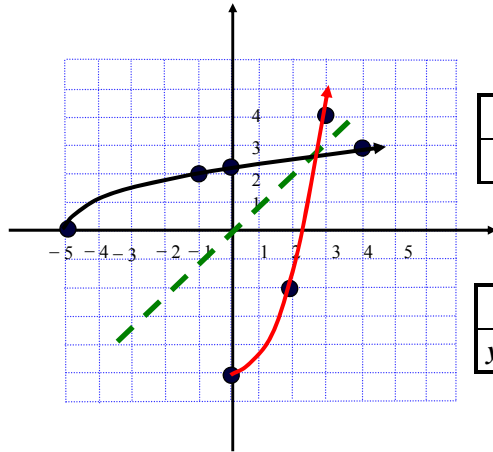
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Ex 5 Sketch the graph of f^{-1} using the graph $f(x) = \sqrt{x+5}$

Sol



Select some points on f

x	-5	-1	0	4
$y=f(x)$	0	2	2.5	3

points on f^{-1}

x	0	2	2.5	3
$y=f^{-1}(x)$	-5	-1	0	4

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Ex 6 Let $f(x) = -x^2 - 3x + k$ such that $f^{-1}(x)$ exists. If $f^{-1}(2) = 3$, then find k .

Sol

$$\text{If } f^{-1}(2) = 3, \text{ then } f(3) = 2$$

$$2 = -(3)^2 - 3(3) + k \quad \text{Thus, } k = 20$$

Remember:

$$\begin{aligned} y &= f(x) \\ \Leftrightarrow \\ x &= f^{-1}(y) \end{aligned}$$

Ex 7 If $f(x) = \frac{2x+1}{x-1}$, then find $(f \circ f^{-1})(5) + f^{-1}(1)$

Sol

$$\begin{aligned} &(f \circ f^{-1})(5) + f^{-1}(1) \\ &\quad \swarrow \quad \searrow \\ &= 5 \quad \quad \quad \frac{2x+1}{x-1} = 1 \Rightarrow x = -2 = f^{-1}(1) \end{aligned}$$

$$\text{Thus, } (f \circ f^{-1})(5) + f^{-1}(1) = 5 - 2 = 3$$

Is 5 in $D_{f^{-1}}$?

equivalently,

Is 5 in R_f ?

$$5 = \frac{2x+1}{x-1}$$

$$x = 2$$

So, yes

$$5 \in R_f = D_{f^{-1}}$$

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Ex 8 If $f^{-1}(x) = 2 + \sqrt{x-1}, x \geq 1$ then find $f(4)$

Sol

Let $f(4) = y$, then $f^{-1}(y) = 4$

Remember:

$$\begin{array}{c} y = f(x) \\ \Leftrightarrow \\ x = f^{-1}(y) \end{array}$$

$$\Rightarrow 4 = 2 + \sqrt{y-1}, y \geq 1 \quad \text{Solve this radical equation}$$

$$\Rightarrow 2 = \sqrt{y-1}, y \geq 1$$

$$\Rightarrow 4 = y - 1, y \geq 1 \quad \text{Solve this radical equation}$$

$$\Rightarrow y = 5$$

$$\Rightarrow f(4) = 5$$