3.5 Graphing Rational Functions

Objectives:

In this section, we will learn about:

- Vertical and Horizontal Asymptotes.
- Sign Property of Rational Functions.
- General Graphing Procedure.
- Slant Asymptotes.
- Graph Rational Functions that Have a Common Factor.

Definition:

A **rational function** is a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$,
where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

Ex 1 Lets observe the behavior of the function $f(x) = \dfrac{1}{x}$ as x approaches 0 from right (0^+) or left (0^-).

x	f(x)
2	0.5
1	1
0.5	2
0.1	10
0.01	100
0.001	1000

$f(x) \to +\infty$ as $x \to 0^+$

x	f(x)
– 2	– 0.5
–1	–1
– 0.5	–2
– 0.1	–10
– 0.01	–100
– 0.001	–1000

$f(x) \to -\infty$ as $x \to 0^-$

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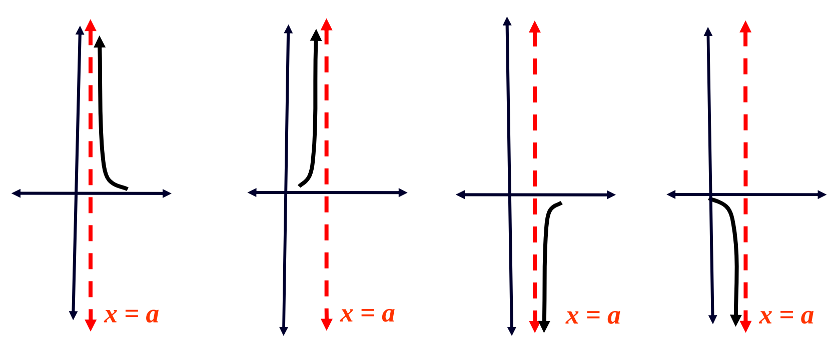
Definition

If for the graph of a function f, $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to a$ from either, left or right, then the line $\boldsymbol{x = a}$ is a **vertical asymptote** **(خط تقارب عمودي)** to the graph of f.



$f(x) \to +\infty$ as $x \to a^+$

$f(x) \to +\infty$ as $x \to a^-$

$f(x) \to -\infty$ as $x \to a^+$

$f(x) \to -\infty$ as $x \to a^-$

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Finding Vertical Asymptotes for Rational Functions

The graph of a rational function $\frac{P(x)}{Q(x)}$, where *P (x) and Q(x)* have **no common factor**, has a vertical asymptote at $x = a$ for any value of a for which $Q(a) = 0$.

Ex 2 Find the vertical asymptotes of the graph of the following functions

a) $f(x) = \dfrac{1}{x^2 + 4x - 5}$

Sol

P (x) and Q(x) have no **common factor(s)**

Do *P (x) and Q(x)* have **common factor(s) ?** If yes **remove** from the equation **and** from the graph.

Set the **denominator** equal to **zero** and solve for *x*,

$$x^2 + 4x - 5 = 0$$
$$(x - 1)(x + 5) = 0$$

Therefore, the graph of f has vertical asymptotes at $x = 1$ and $x = -5$

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b) $g(x) = \dfrac{x+1}{x^2 - 1}$

Sol *P (x) and Q(x)* have **common factor(s)** since

Do *P (x) and Q(x)* have **common factor (s)?** If yes **remove** from the equation **and** from the graph.

$$g(x) = \frac{\cancel{x+1}}{(x-1)\cancel{(x+1)}}$$

$$g(x) = \frac{1}{(x-1)} \quad , x \neq -1$$

Missing point on the graph

Now, set the **denominator** equal to **zero** and solve for *x*,

$$x - 1 = 0$$
$$x = 1$$

Therefore, the graph of f has a vertical asymptote at $x = 1$

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c) $g(x)=\dfrac{x+1}{x^2+1}$

Sol *P (x) and Q(x)* have no **common factor(s)** ?

Do *P (x) and Q(x)* have **common factor (s)?** If yes **remove** from the equation and from the graph.

Set the **denominator** equal to zero and solve for x.

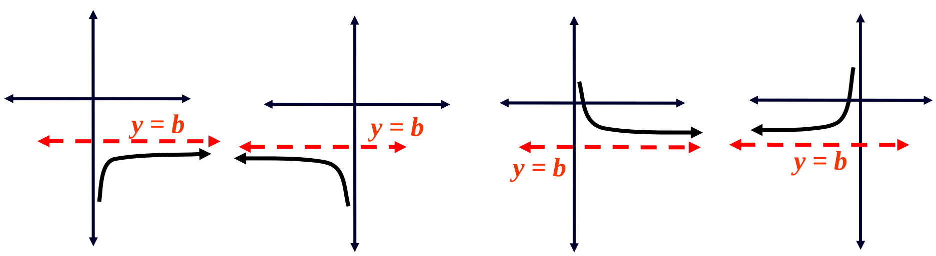
$$x^2 + 1 = 0$$

The quadratic equation $x^2 + 1 = 0$ has no real solutions ?

Therefore, the graph of f has no vertical asymptote.

Definition

If for the graph of a function f, $f(x) \to b$ as $x \to +\infty$ or as $x \to -\infty$, then the line $y = b$ is a **horizontal asymptote (خط تقارب أفقي)** to the graph of f.



$f(x) \to b$ as $x \to +\infty$

$f(x) \to b$ as $x \to -\infty$

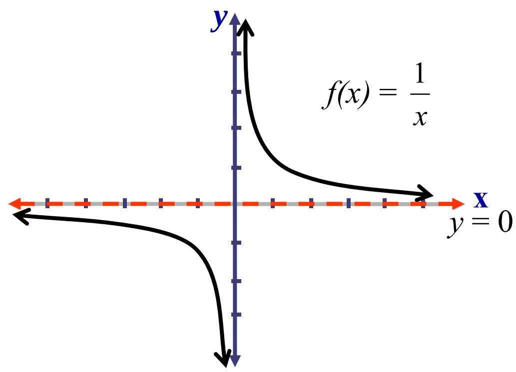
$f(x) \to b$ as $x \to +\infty$

$f(x) \to b$ as $x \to -\infty$

Ex 3 The line $y = 0$ is a horizontal asymptote of the graph of the function $f(x) = \frac{1}{x}$.

As x becomes unbounded positively, $f(x)$ **approaches** zero from above; therefore, the line $y = 0$ is a horizontal asymptote of the graph of f.

x	$f(x)$
10	0.1
100	0.01
1000	**0.001**
0	$\pm\infty$
–10	– 0.1
–100	– 0.01
– 1000	**– 0.001**



Finding Horizontal Asymptotes for Rational Functions

Given a rational function $f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0}{b_n x^n + b_{m-1} x^{m-1} + \ldots + b_0}$

where $P(x)$ *and* $Q(x)$ have **no common factor .**

- If $m > n$, then there are **no** horizontal asymptotes.
- If $m < n$, then $\boldsymbol{y = 0}$ is a horizontal asymptote.
- If $m = n$, then $y = \dfrac{a_m}{b_n}$ is a horizontal asymptote.

Ex 4 Find the horizontal asymptote to the graph of each of the following rational function:

a) $f(x) = \dfrac{2x+1}{x^2 - 3x - 4}$ b) $g(x) = \dfrac{3x^2 - 1}{-2x^2 + x - 1}$ c) $h(x) = \dfrac{-2x^5}{3x^2 + 1}$

$a)\ f(x) = \dfrac{2x+1}{x^2-3x-4}$

Do *P (x) and Q(x)* have **common factor(s) ?** If yes **remove** from the equation and from the graph.

Sol $f(x) = \dfrac{2x+1}{(x-4)(x+1)}$

P (x) and Q(x) have **no common factor.**

Since **Degree *of P (x)* < Degree of *Q(x)***

then $\mathbf{y = 0}$ is a horizontal asymptote to the graph of f

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$b)\ g(x) = \dfrac{3x^2-1}{-2x^2+x-1}$

Do *P (x) and Q(x)* have **common factor ?** If yes **remove** from the equation and from the graph

Sol

P (x) and *Q(x)* have **no common factor(s) ?**

Since **Degree of *P (x)* = Degree of *Q(x)***

, then $y = \dfrac{3}{-2}$ is a horizontal asymptote to the graph of f

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c) $h(x) = \dfrac{-2x^5}{3x^2+1}$

Do *P (x) and Q(x)* have **common factor ?**
If yes **remove** from the equation and from the graph.

Sol

P (x) and Q(x) have **no common factor?**

Since **Degree of *P (x)* > Degree of *Q(x)***

, then there is **no** horizontal asymptote to the graph of *f.*

Slant Asymptotes

The rational function given by $f(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ have **no common factors**, has *a* **slant asymptote** if the **degree** of the polynomial $P(x)$ in the **numerator** is **one greater than** the **degree** of the polynomial $Q(x)$ in the **denominator**.

Finding the Slant Asymptote:

Using long division $f(x) = \dfrac{P(x)}{Q(x)} = (mx+b) + \dfrac{r(x)}{Q(x)}$,

slant asymptote
$y = m\,x + b$

Ex 5 Find the slant asymptote to the graph of $f(x)=\dfrac{4x^3+7x^2+22x-8}{x^2+2x+5}$

Sol *P (x) and Q(x)* have **no common factor?**

Using long division:

$$\begin{array}{r|l} & \quad 4x-1 \\ \hline x^2+2x+5 & 4x^3+7x^2+22x-8 \\ & 4x^3+8x^2+20x \\ \hline & \quad -x^2+2x-8 \\ & \quad -x^2-2x-5 \\ \hline & \qquad 4x-3 \end{array}$$

Do *P (x) and Q(x)* have **common factor ?** If yes **remove** from the equation **and** from the graph.

Therefore the slant asymptote is given by $y=4x-1$

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Practice Determine all asymptotes for the graph of the following functions.

a) $F(x)=\dfrac{2x^2-4x+5}{x-3}$

b) $G(x)=\dfrac{2x(x^2-4x-12)}{x^3-36x}$

c) $h(x)=\dfrac{x^2-1}{x^3-1}$

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Practice Give an example of a rational function without any asymptote.

Practice If $y = -3$ is a horizontal asymptote for the graph of $F(x) = \dfrac{ax^2 + x + 2}{2x^2 - x - 1}$, then find the vertical asymptote(s).

A Sign Property of Rational Function

The **zeros** (***x*-intercepts**) and **vertical asymptotes** of a rational function F divide the ***x*-axis into intervals**. In each interval:

- *If* $F(x)$ is **positive,** then its graph **lies above the *x*-axis** for all x in the interval.
- If $F(x)$ is **negative** , then its graph **lies below the *x*-axis** for all x in the interval.

General Procedure for Graphing Rational Functions that Have No Common Factors:

1) Draw all asymptotes(if any) as dashed lines observing:
    1) The behavior of the function near the asymptote.
    2) The graph may **not** intersect the vertical asymptote.
    3) The graph **may intersect** the horizontal or slant asymptotes.
2) Do sign test for the function, to see whether the graph lies above or below the x-axis.
3) Find all x-intercepts and y-intercepts.
4) Observe all types of symmetry.
5) Use additional points as needed.

Ex 6 Sketch the graph of $f(x)=\dfrac{x+2}{x-2}$

Sol

<u>**Asymptotes**</u> *P (x) and Q(x)* have **no common factor?**

V.A.: $x = 2$

H.A.: $y = 1$ S.A. : NO.

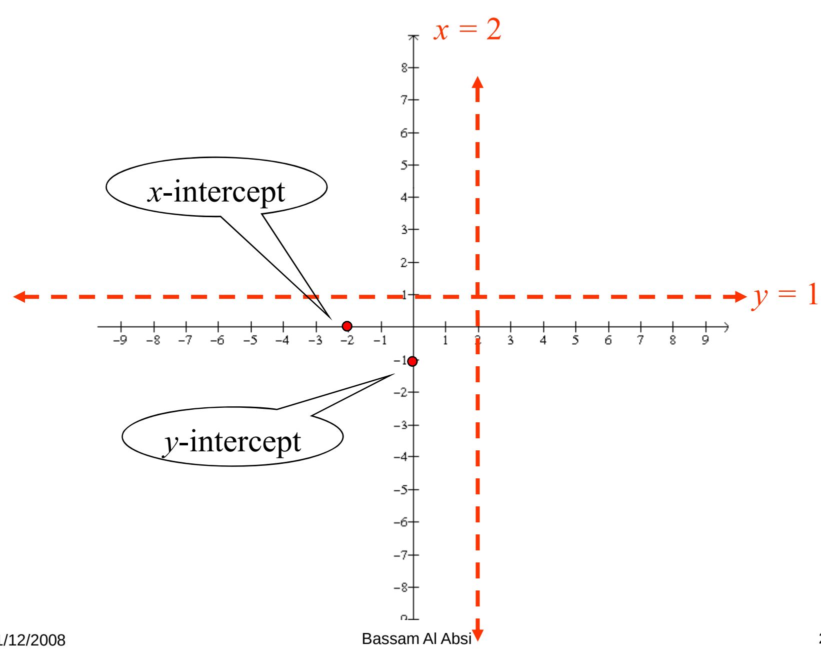
Check if the graph intersects the H.A.

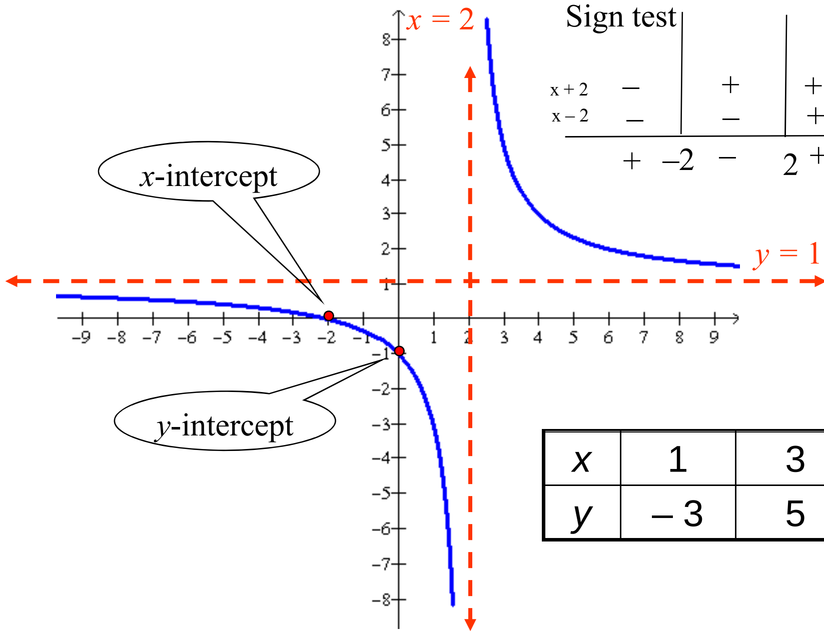
Solve the equation: $f(x)=1 \Rightarrow x+2=x-2 \Rightarrow 2=-2$

Therefore, the graph does not intersect the H.A.

<u>**Intercepts**</u>

- x-intercept: put $y=0 \Rightarrow x+2=0 \Rightarrow x=-2 \Rightarrow (-2,0)$
- y-intercept: put $x=0 \Rightarrow y=-1 \Rightarrow (0,-1)$





Sign test

x + 2	−	+	+
x − 2	−	−	+
	+ −2	− 2	+

x	1	3
y	– 3	5

Ex 7 Sketch the graph of the following functions:

$$a)\quad f(x) = \frac{x^2+1}{x^2+x-2}$$

Sol

Asymptotes $P(x)$ *and* $Q(x)$ have **no common factor?**

V.A.: $x^2 + x - 2 = 0$, implies $x = -2,\ 1$

H.A.: $y = 1$ S.A. : NO

Check if the graph intersects the H.A.

Solve the equation: $f(x) = 1$, implies $x^2 + x - 2 = x^2 + 1 \quad \rightarrow x = 3$

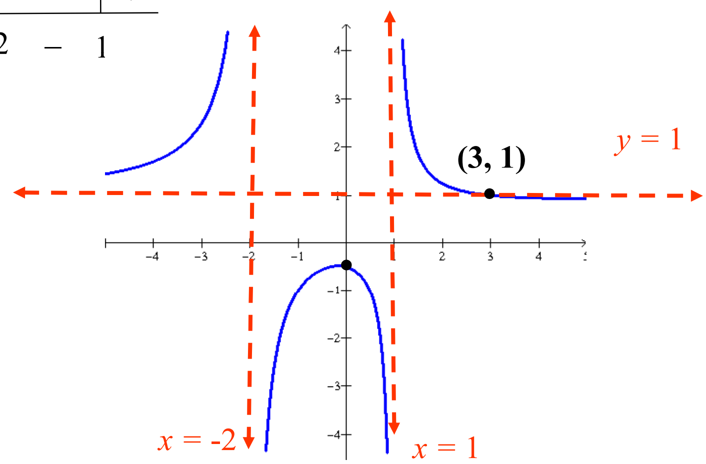
Therefore, the graph intersects the H.A. at (3, **1**)

Intercepts

- x-intercept: put $y = 0 \Rightarrow x^2 + 1 = 0 \Rightarrow No\ \ x-$intercept.
- y-intercept: put $x = 0 \Rightarrow y = \frac{-1}{2} \Rightarrow (0, \frac{-1}{2})$

Sign test

x^2+1	+	+	+
x^2+x-2	+	−	+
	+	−2 −	1



$$b)\ g(x)=\frac{x^2+x}{x-1}$$

Sol

<u>Asymptotes</u> *P (x) and Q(x)* have **no common factor?**

V.A.: $x-1=0$, implies $x=1$

H.A.: NO S. A. : $y=x+2$ **?**

<u>Check if the graph intersects the S.A.</u>

Solve the equation $f(x)=x+2$, implies $x^2+x-2=x^2+x \longrightarrow 0=-2$

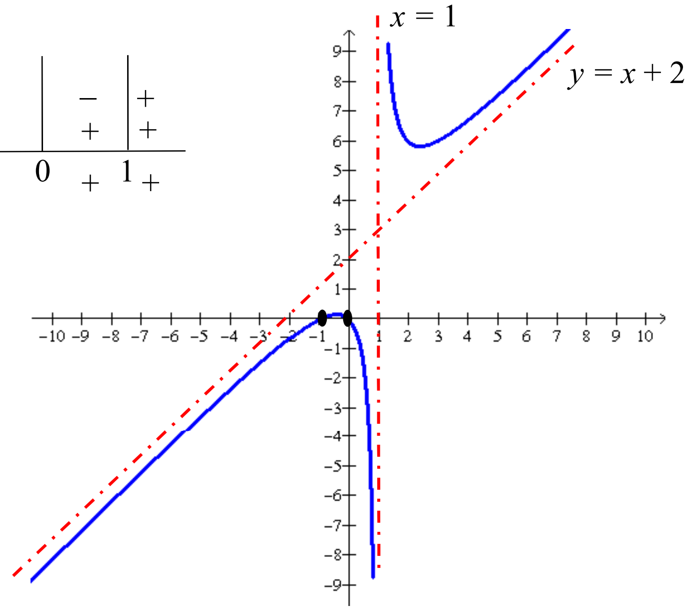
Therefore, the graph **does not** intersect the S.A.

<u>Intercepts</u>

- x-intercept: put $y=0 \Rightarrow x^2+x=0 \Rightarrow x=0,-1 \Rightarrow (0,0),(-1,0)$
- y-intercept: put $x=0 \Rightarrow y=0 \Rightarrow (0,0)$

Sign test

$x-1$	$-$	$-$	$-$	$+$
x^2+x	$+$	$-$	$+$	$+$
	$-$	-1 $+$	0 $+$	1 $+$



$c)\ h(x) = \dfrac{2x^2 - 8}{x^2 + 2}$

Sol

Asymptotes *P (x) and Q(x)* have **no common factor?**

V.A.: NO H.A.: $y = 2$ S. A. : NO

Check if the graph intersects the H.A.

Solve the equation: $f(x) = 2,$ implies $2x^2 - 8 = 2x^2 + 4 \rightarrow -8 = 0$

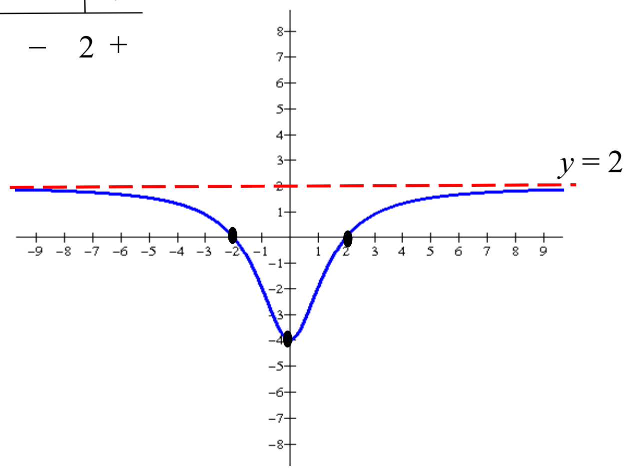
Therefore, the graph does not intersect the H.A.

Intercepts

- x-intercept: put $y = 0 \Rightarrow 2x^2 - 8 = 0 \Rightarrow x = -2, 2 \Rightarrow (-2,0), (2,0)$
- y-intercept: put $x = 0 \Rightarrow y = -4 \Rightarrow (0, -4)$

Sign test

x^2+2	+	+	+
$2x^2-8$	+	−	+
	+ -2	−	2 +



$d)\ k(x)=\dfrac{3x^2-3}{x^2-x-2}$ *P (x) and Q(x)* have **no common factor?**

Sol

$k(x)=\dfrac{3(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(x-2)}$ $\Rightarrow k(x)=\dfrac{3(x-1)}{(x-2)},\ x\neq -1$

How is it obtained ?

The graph has a **missing point(Hole)** at (−1, 2)

Asymptotes

V.A.: $x=2$ H.A.: $y=3$ S. A. : NO

Check if the graph intersects the H.A.

Solve the equation: $f(x)=3$, implies $x-1=x-2 \ \rightarrow -1=-8$

Therefore, the graph does **not** intersect the H.A.

Intercepts

$$k(x)=\frac{3(x-1)}{(x-2)},\quad x\neq -1$$

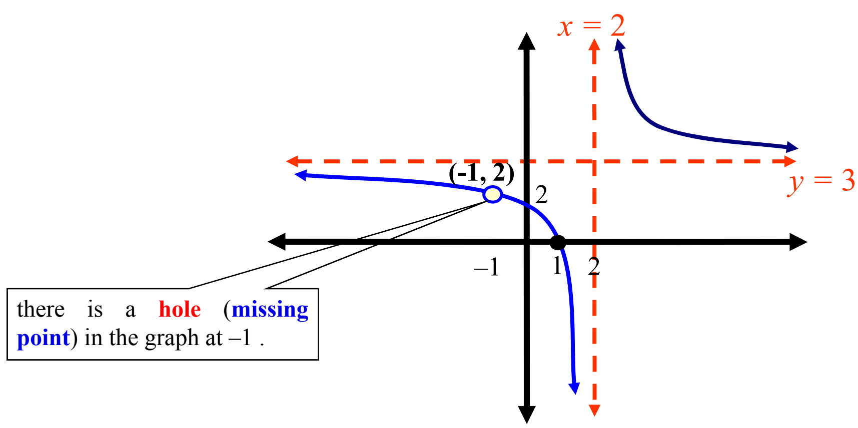
- x-intercept: put $y=0 \Rightarrow 3(x-1)=0$

  $\Rightarrow x=1 \Rightarrow (1,0)$

- y-intercept: put $x=0 \Rightarrow y=\frac{3}{2} \Rightarrow (0,\frac{3}{2})$

Sign test

$x-1$	$-$	$+$	$+$
$x-2$	$-$	$-$	$+$
	$+$	1 $-$	2 $+$



Ex 8 Determine the point at which the graph of the following function intersects the slant asymptote.

$$F(x) = \frac{3x^3 + 2x^2 - 8x - 12}{x^2 + 4}$$

Sol

Using the long division

$$\begin{array}{r|l} & 3x + 2 \\ \hline x^2 + 0x + 4 & 3x^3 + 2x^2 - 8x - 12 \\ & \underline{3x^3 + 0x^2 + 12x} \\ & \quad 2x^2 - 20x - 12 \\ & \quad \underline{2x^2 + 0x + 8} \\ & \qquad -20x - 20 \end{array}$$

Now, check if the following equation has solution (s) ?

$$\frac{3x^3 + 2x^2 - 8x - 12}{x^2 + 4} = 3x + 2$$

$$3x^3 + 2x^2 + 12x + 8 = 3x^3 + 2x^2 - 8x - 12$$

$$\Rightarrow x = -1$$

$$and \quad y = -1$$

How is it obtained ?

The graph of F intersects the slant asymptote at the point (1, – 3)