# 3.5 Graphing Rational Functions



## Objectives:

In this section, we will learn about:

- Vertical and Horizontal Asymptotes.
- Sign Property of Rational Functions.
- General Graphing Procedure.
- Slant Asymptotes.
- Graph Rational Functions that Have a Common Factor.

## Definition:

A **rational function** is a function of the form $f(x) = \dfrac{P(x)}{Q(x)}$ ,
where $P(x)$ and $Q(x)$ are polynomials and $Q(x) \neq 0$.

**Ex 1** Lets observe the behavior of the function $f(x) = \dfrac{1}{x}$ as x approaches 0 from right ($0^+$) or left ($0^-$).

| *x* | *f(x)* |
|---|---|
| 2 | 0.5 |
| 1 | 1 |
| 0.5 | 2 |
| 0.1 | 10 |
| 0.01 | 100 |
| 0.001 | 1000 |

$f(x) \to +\infty$ as $x \to 0^+$

| *x* | *f(x)* |
|---|---|
| – 2 | – 0.5 |
| –1 | –1 |
| – 0.5 | –2 |
| – 0.1 | –10 |
| – 0.01 | –100 |
| – 0.001 | –1000 |

$f(x) \to -\infty$ as $x \to 0^-$

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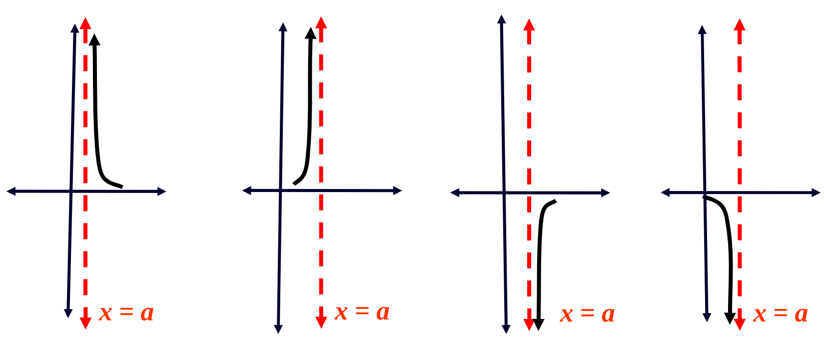
## Definition

If for the graph of a function $f$, $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to a$ from either, left or right, then the line $\boldsymbol{x = a}$ is a **vertical asymptote** **(خط تقارب عمودي)** to the graph of $f$.




$f(x) \to +\infty$ as $x \to a^+$

$f(x) \to +\infty$ as $x \to a^-$

$f(x) \to -\infty$ as $x \to a^+$

$f(x) \to -\infty$ as $x \to a^-$

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## Finding Vertical Asymptotes for Rational Functions

The graph of a rational function $\frac{P(x)}{Q(x)}$, where *P (x) and Q(x)* have **no common factor**, has a vertical asymptote at $x = a$ for any value of $a$ for which $Q(a) = 0$ .

**Ex 2** Find the vertical asymptotes of the graph of the following functions

a) $f(x) = \dfrac{1}{x^2 + 4x - 5}$

**Sol**

*P (x) and Q(x)* have no **common factor(s)**

Set the **denominator** equal to **zero** and solve for *x*,

$$x^2 + 4x - 5 = 0$$
$$(x - 1)(x + 5) = 0$$

Do *P (x) and Q(x)* have **common factor(s) ?** If yes **remove** from the equation **and** from the graph.

Therefore, the graph of $f$ has vertical asymptotes at $x = 1$ and $x = -5$

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b) $g(x) = \dfrac{x+1}{x^2 - 1}$

**Sol** *P (x) and Q(x)* have **common factor(s)** since

$$g(x) = \frac{\cancel{x+1}}{(x-1)\cancel{(x+1)}}$$

$$g(x) = \frac{1}{(x-1)} \quad , x \neq -1$$

Missing point on the graph

Do *P (x) and Q(x)* have **common factor (s)?** If yes **remove** from the equation **and** from the graph.

Now, set the **denominator** equal to **zero** and solve for *x*,

$$x - 1 = 0$$
$$x = 1$$

Therefore, the graph of $f$ has a vertical asymptote at $x = 1$

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$c)\ g(x) = \dfrac{x+1}{x^2+1}$

**Sol** *P (x) and Q(x)* have no **common factor(s)** ?

Do *P (x) and Q(x)* have **common factor (s)?** If yes **remove** from the equation and from the graph.

Set the **denominator** equal to zero and solve for $x$.

$$x^2 + 1 = 0$$

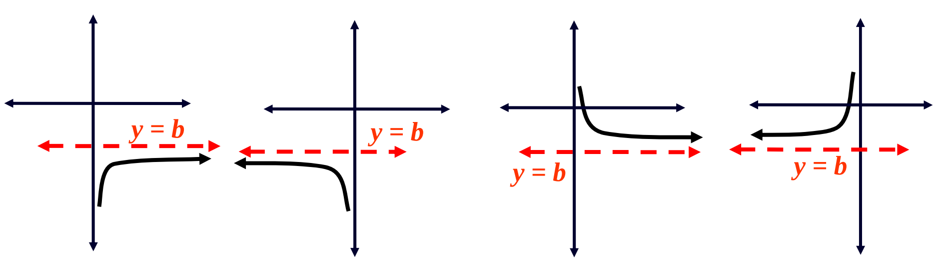
The quadratic equation $x^2 + 1 = 0$ has no real solutions ?

Therefore, the graph of $f$ has no vertical asymptote.



**Definition**

If for the graph of a function $f$, $f(x) \to b$ as $x \to +\infty$ or as $x \to -\infty$, then the line $y = b$ is a **horizontal asymptote** (خط تقارب أفقي) to the graph of $f$.




$f(x) \to b$ as $x \to +\infty$

$f(x) \to b$ as $x \to -\infty$

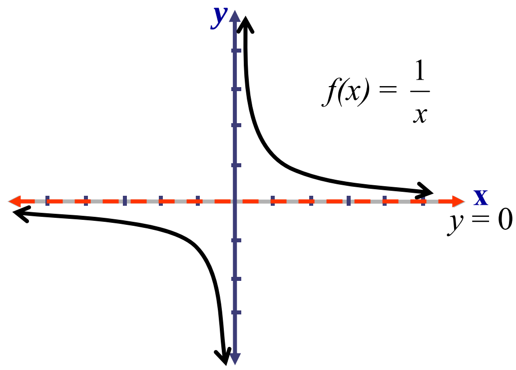
$f(x) \to b$ as $x \to +\infty$

$f(x) \to b$ as $x \to -\infty$

**Ex 3** The line $y = 0$ is a horizontal asymptote of the graph of the function $f(x) = \frac{1}{x}$ .

As $x$ becomes unbounded positively, $f(x)$ **approaches** zero from above; therefore, the line $y = 0$ is a horizontal asymptote of the graph of $f$.

| x | f(x) |
|---|---|
| 10 | 0.1 |
| 100 | 0.01 |
| **1000** | **0.001** |
| 0 | $\pm\infty$ |
| –10 | – 0.1 |
| –100 | – 0.01 |
| **– 1000** | **– 0.001** |






## Finding Horizontal Asymptotes for Rational Functions

Given a rational function $f(x) = \dfrac{P(x)}{Q(x)} = \dfrac{a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0}{b_n x^n + b_{m-1} x^{m-1} + \ldots + b_0}$

where $P(x)$ and $Q(x)$ have **no common factor .**

- If $m > n$, then there are **no** horizontal asymptotes.
- If $m < n$, then $\mathbf{y = 0}$ is a horizontal asymptote.
- If $m = n$, then $y = \frac{a_m}{b_n}$ is a horizontal asymptote.

**Ex 4** Find the horizontal asymptote to the graph of each of the following rational function:

a) $f(x) = \dfrac{2x+1}{x^2-3x-4}$ b) $g(x) = \dfrac{3x^2-1}{-2x^2+x-1}$ c) $h(x) = \dfrac{-2x^5}{3x^2+1}$

Do *P (x) and Q(x)* have **common factor(s) ?** If yes **remove** from the equation and from the graph.

$$a)\ \ f(x)=\frac{2x+1}{x^2-3x-4}$$

**Sol** $f(x)=\dfrac{2x+1}{(x-4)(x+1)}$

*P (x) and Q(x)* have **no common factor.**

Since **Degree *of P (x)* < Degree of *Q(x)***

then $\boldsymbol{y=0}$ is a horizontal asymptote to the graph of $f$



Do *P (x) and Q(x)* have **common factor ?** If yes **remove** from the equation and from the graph

$$b)\ \ g(x)=\frac{3x^2-1}{-2x^2+x-1}$$

**Sol**

*P (x)* and *Q(x)* have **no common factor(s) ?**

Since **Degree of *P (x)* = Degree of *Q(x)***

, then $y=\dfrac{3}{-2}$ is a horizontal asymptote to the graph of $f$

c) $h(x) = \dfrac{-2x^5}{3x^2+1}$

Do *P (x) and Q(x)* have **common factor ?**
If yes **remove** from the equation and from the graph.

**Sol**

*P (x) and Q(x)* have **no common factor?**

Since **Degree of *P (x)* > Degree of *Q(x)***

, then there is **no** horizontal asymptote to the graph of *f.*



## Slant Asymptotes

The rational function given by $f(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ have **no common factors**, has *a* **slant asymptote** if the **degree** of the polynomial $P(x)$ in the **numerator** is **one greater than** the **degree** of the polynomial $Q(x)$ in the **denominator**.

## Finding the Slant Asymptote:

Using long division $f(x) = \dfrac{P(x)}{Q(x)} = (mx+b) + \dfrac{r(x)}{Q(x)}$,

slant asymptote
$y = m\,x + b$

**Ex 5** Find the slant asymptote to the graph of $f(x)=\dfrac{4x^3+7x^2+22x-8}{x^2+2x+5}$

**Sol** *P (x) and Q(x)* have **no common factor?**

Using long division:

$$\begin{array}{r|l} & 4x-1 \\ \hline x^2+2x+5 & 4x^3+7x^2+22x-8 \\ & 4x^3+8x^2+20x \\ \hline & -x^2+2x-8 \\ & -x^2-2x-5 \\ \hline & 4x-3 \end{array}$$

Do *P (x) and Q(x)* have **common factor ?** If yes **remove** from the equation **and** from the graph.

Therefore the slant asymptote is given by $y=4x-1$



**Practice** Determine all asymptotes for the graph of the following functions.

a) $F(x)=\dfrac{2x^2-4x+5}{x-3}$

b) $G(x)=\dfrac{2x(x^2-4x-12)}{x^3-36x}$

c) $h(x)=\dfrac{x^2-1}{x^3-1}$

**Practice** Give an example of a rational function without any asymptote.

**Practice** If $y = -3$ is a horizontal asymptote for the graph of $F(x)=\dfrac{ax^2+x+2}{2x^2-x-1}$, then find the vertical asymptote(s).

## A Sign Property of Rational Function

The **zeros** (***x*-intercepts**) and **vertical asymptotes** of a rational function *F* divide the ***x*-axis into intervals**. In each interval:

- *If F*($x$) is **positive,** then its graph **lies above the *x*-axis** for all $x$ in the interval.
- If *F*($x$) is **negative** , then its graph **lies below the *x*-axis** for all $x$ in the interval.



## General Procedure for Graphing Rational Functions that Have No Common Factors:

1) Draw all asymptotes( if any) as dashed lines observing:
    1) The behavior of the function near the asymptote.
    2) The graph may **not** intersect the vertical asymptote.
    3) The graph **may intersect** the horizontal or slant asymptotes.
2) Do sign test for the function, to see whether the graph lies above or below the $x$-axis.
3) Find all $x$-intercepts and $y$-intercepts.
4) Observe all types of symmetry.
5) Use additional points as needed.

**Ex 6** Sketch the graph of $f(x) = \dfrac{x+2}{x-2}$

**Sol**

**<u>Asymptotes</u>** *P (x) and Q(x)* have **no common factor?**

V.A.: $x = 2$

H.A.: $y = 1$ S.A. : NO.

**Check if the graph intersects the H.A.**

Solve the equation: $f(x) = 1 \Rightarrow x + 2 = x - 2 \Rightarrow 2 = -2$

Therefore, the graph does not intersect the H.A.

**<u>Intercepts</u>**

- $x$-intercept: put $y = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2 \Rightarrow (-2, 0)$
- $y$-intercept: put $x = 0 \Rightarrow y = -1 \Rightarrow (0, -1)$

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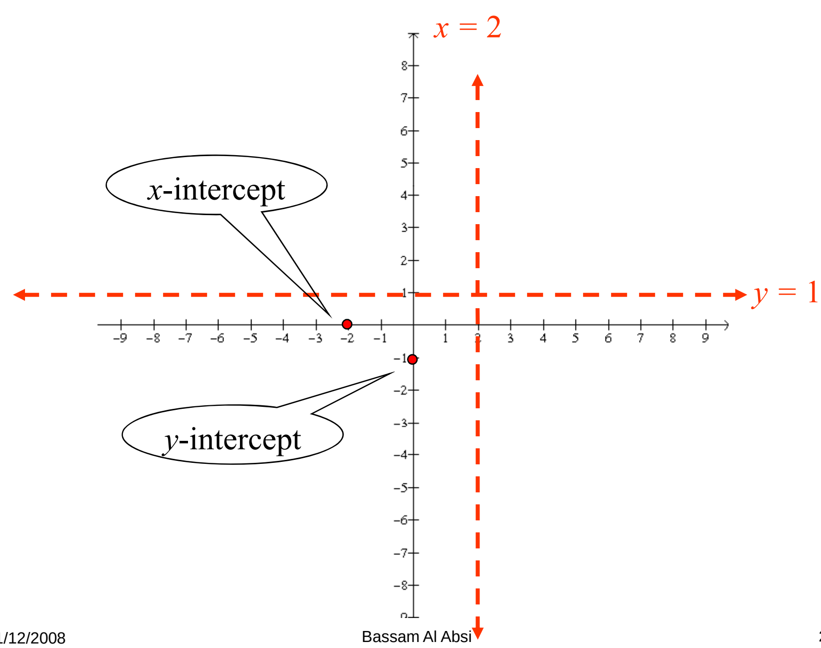


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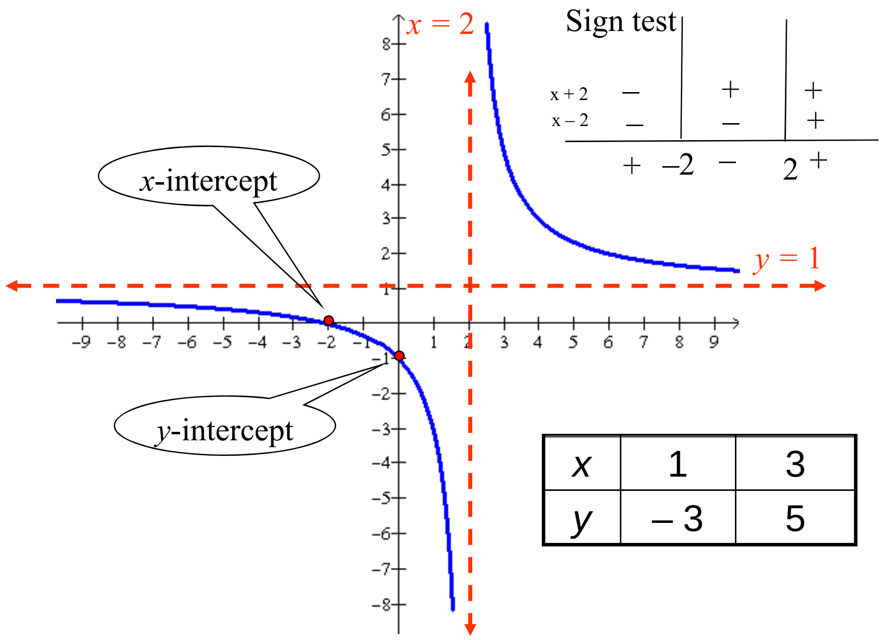



| $x$ | 1 | 3 |
|---|---|---|
| $y$ | – 3 | 5 |



**Ex 7** Sketch the graph of the following functions:

$a)\quad f(x) = \dfrac{x^2+1}{x^2+x-2}$

**Sol**

**Asymptotes** $\quad P(x)$ *and* $Q(x)$ have **no common factor?**

V.A.: $x^2 + x - 2 = 0$ , implies $x = -2\,,\ 1$

H.A.: $y = 1$ S.A. : NO

**Check if the graph intersects the H.A.**

Solve the equation: $f(x) = 1$ , implies $x^2 + x - 2 = x^2 + 1 \ \rightarrow x = 3$

Therefore, the graph intersects the H.A. at ( 3, 1)

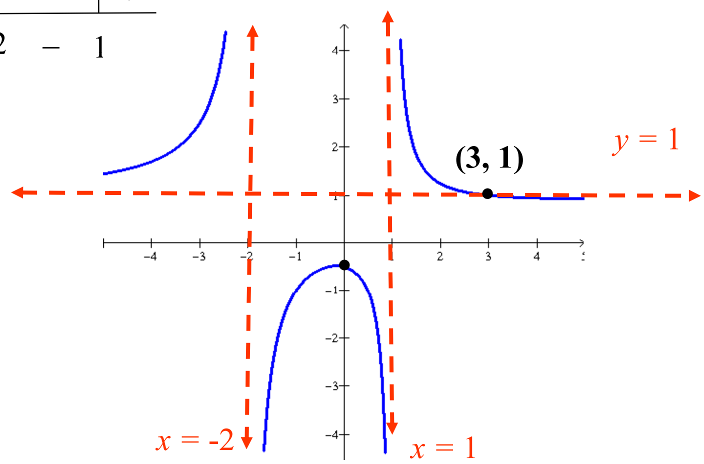
**Intercepts**

- $x$-intercept: put $y = 0 \Rightarrow x^2 + 1 = 0 \Rightarrow$ *No* $x-$intercept.
- $y$-intercept: put $x = 0 \Rightarrow y = \dfrac{-1}{2} \Rightarrow (0, \dfrac{-1}{2})$

Sign test

| | | | |
|---|---|---|---|
| $x^2+1$ | $+$ | $+$ | $+$ |
| $x^2+x-2$ | $+$ | $-$ | $+$ |

$+ \quad -2 \quad - \quad 1$






$b)\ g(x)=\dfrac{x^2+x}{x-1}$

**Sol**

**Asymptotes** *P (x) and Q(x)* have **no common factor?**

V.A.: $x-1=0$ , implies $x=1$

H.A.: NO S. A. : $y=x+2$ **?**

**Check if the graph intersects the S.A.**

Solve the equation $f(x)=x+2$ , implies $x^2+x-2=x^2+x \longrightarrow 0=-2$

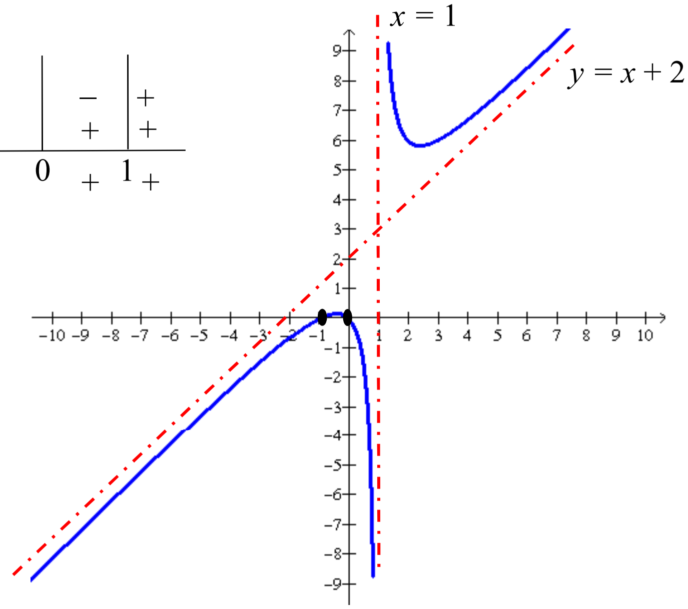
Therefore, the graph **does not** intersect the S.A.

**Intercepts**

- $x$-intercept: put $y=0 \Rightarrow x^2+x=0 \Rightarrow x=0,-1 \Rightarrow (0,0),(-1,0)$
- $y$-intercept: put $x=0 \Rightarrow y=0 \Rightarrow (0,0)$

Sign test

| | $(-\infty,-1)$ | $(-1,0)$ | $(0,1)$ | $(1,\infty)$ |
|---|---|---|---|---|
| $x-1$ | $-$ | $-$ | $-$ | $+$ |
| $x^2+x$ | $+$ | $-$ | $+$ | $+$ |
| | $-$ | $+$ | $+$ | $+$ |

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$$c)\ h(x) = \frac{2x^2 - 8}{x^2 + 2}$$

**Sol**

**Asymptotes** *P (x) and Q(x)* have **no common factor?**

V.A.: NO H.A.: $y = 2$ S. A. : NO

**Check if the graph intersects the H.A.**

Solve the equation: $f(x) = 2$, implies $2x^2 - 8 = 2x^2 + 4 \rightarrow -8 = 0$

Therefore, the graph does not intersect the H.A.

**Intercepts**

- $x$-intercept: put $y = 0 \Rightarrow 2x^2 - 8 = 0 \Rightarrow x = -2, 2 \Rightarrow (-2,0), (2,0)$
- $y$-intercept: put $x = 0 \Rightarrow y = -4 \Rightarrow (0,-4)$

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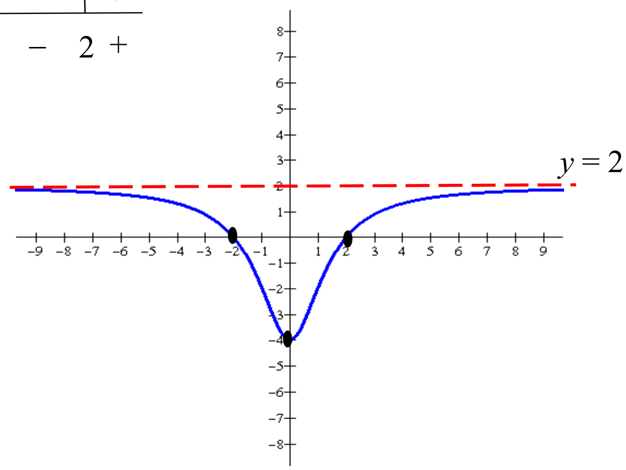
Sign test

| | | | |
|---|---|---|---|
| $x^2+2$ | + | + | + |
| $2x^2-8$ | + | − | + |
| | + $-2$ | − | $2$ + |






$d)\ k(x)=\dfrac{3x^2-3}{x^2-x-2}$ *P (x) and Q(x)* have **no common factor?**

**Sol**

$$k(x)=\frac{3(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(x-2)}\qquad \Rightarrow k(x)=\frac{3(x-1)}{(x-2)},\ x\neq -1$$

How is it obtained ?

The graph has a **missing point( Hole)** at $(-1, 2)$

**Asymptotes**

V.A.: $x=2$ H.A.: $y=3$ S. A. : NO

**Check if the graph intersects the H.A.**

Solve the equation: $f(x)=3$ , implies $x-1=x-2 \rightarrow -1=-8$

Therefore, the graph does **not** intersect the H.A.

## Intercepts

$$k(x)=\frac{3(x-1)}{(x-2)},\quad x\neq -1$$

- $x$-intercept: put $y=0 \Rightarrow 3(x-1)=0$

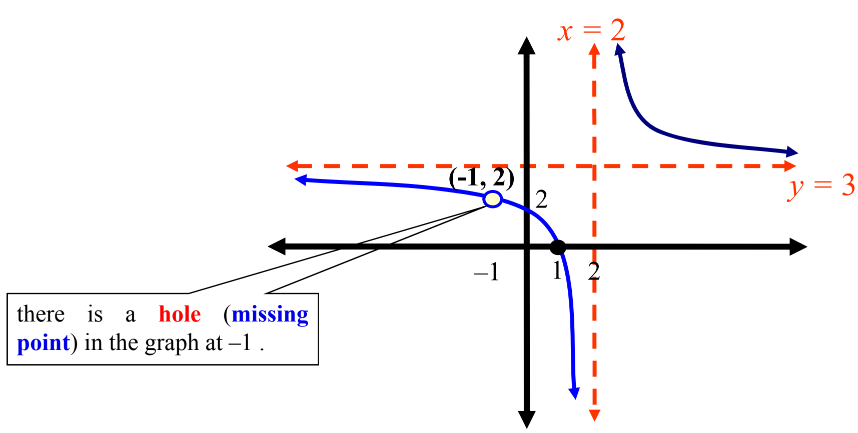
  $\Rightarrow x=1 \Rightarrow (1,0)$

- $y$-intercept: put $x=0 \Rightarrow y=\frac{3}{2} \Rightarrow (0,\frac{3}{2})$

Sign test

| | | | |
|---|---|---|---|
| $x-1$ | $-$ | $+$ | $+$ |
| $x-2$ | $-$ | $-$ | $+$ |
| | $+$ 1 | $-$ 2 | $+$ |





**Ex 8** Determine the point at which the graph of the following function intersects the slant asymptote.

$$F(x) = \frac{3x^3 + 2x^2 - 8x - 12}{x^2 + 4}$$

**Sol**

Using the long division

$$\begin{array}{r|l} & 3x + 2 \\ \hline x^2 + 0x + 4 & 3x^3 + 2x^2 - 8x - 12 \\ & \underline{3x^3 + 0x^2 + 12x} \\ & \quad 2x^2 - 20x - 12 \\ & \quad \underline{2x^2 + 0x + 8} \\ & \qquad -20x - 20 \end{array}$$

Now, check if the following equation has solution (s) ?

$$\frac{3x^3 + 2x^2 - 8x - 12}{x^2 + 4} = 3x + 2$$

$$3x^3 + 2x^2 + 12x + 8 = 3x^3 + 2x^2 - 8x - 12$$

$$\Rightarrow x = -1$$

$$and \quad y = -1$$

How is it obtained ?

The graph of $F$ intersects the slant asymptote at the point ( 1, – 3)