3.4 THE FUNDAMENTAL THEOREM OF ALGEBRA

Objectives:

In this section, we will learn about:

- The Fundamental Theorem of Algebra.
- The Number of Zeros of a Polynomial Function.
- The Conjugate Zero Theorem.
- Finding a Polynomial Function with Given Zeros.

The Fundamental Theorem of Algebra

If $P(x)$ is a polynomial of degree $n \geq 1$ with complex coefficients, then $P(x)$ has at least one complex zero.

The Linear Factor Theorem

If $P(x)$ is a polynomial of degree $n \geq 1$ with leading coefficients $a_n \neq 0$,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$$

then $P(x)$ has exactly n linear factors

$$P(x) = a_n (x - c_1)(x - c_2) \ldots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex zeros.

Ex 1 Find all the zeros of the polynomial function

$$P(x) = x^4 - 4x^3 + 53x^2 - 196x + 196,$$

and write this polynomial as a product of linear factors.

Sol

Applying the Rational Zero Theorem, $P(x)$ has the following possible rational zeros

$$\pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28, \pm 49, \pm 98, \pm 196$$

Using Descartes' Rule of Sign, P (x) has **no negative real zeros (?)**

Now check the positive candidates zeros **(ONLY ?) ,** starting from the smallest

2	1	−4	53	−196	196
		2	−4	98	−196
	1	−2	49	−98	0

Therefore, 2 is a zero and P (x) is factored as

$$P(x) = (x-2)\left(x^3 - 2x^2 + 49x - 98\right)$$

Now, continue working with the quotient $x^3 - 2x^2 + 49x - 98 = 0$

which has the possible rational zeros $\pm 1, \pm 2, \pm 7, \pm 14, \pm 49, \pm 98$

2	1	−2	49	−98
		2	0	98
	1	0	49	0

So *2 is a zero of* ***multiplicity 2*** *and P* (x) is factored as

$$P(x) = (x-2)^2\left(x^2 + 49\right)$$

Now find the zeros of the reduced polynomial,

$$x^2 + 49 = 0 \quad \Rightarrow x^2 = -49 \quad \Rightarrow x = \pm\sqrt{-49} = \pm 7i$$

The zeros are 2 (multiplicity 2), 7*i*, −7i.

P (x) is factored into the following linear factors,

$$P(x) = (x-2)(x-2)(x-7i)(x+7i)$$

The Conjugate Zero Theorem

If $a + bi$ $(b \neq 0)$ is a complex zero of a polynomial function with ***real coefficients***, then the conjugate $a - bi$ is also a zero of the polynomial function.

Ex2 Find the sum of the remaining zeros of

$P(x) = x^4 - 4x^3 + 14x^2 - 4x + 13$ given that $2 - 3i$ is a zero.

Sol

Using the Conjugate Zero Theorem, $2 - 3i$ is a zero

implies that $2 + 3i$ is also a zero

Now use synthetic division to find the remaining zeros, as follows,

2–3i	1	–4	14	–4	13
		2 – 3i	–13	2 – 3i	–13
	1	–2 – 3i	1	–2 – 3i	0

$P(x)$ factors as

$$P(x) = \left(x - (2 - 3i)\right)\left(x^3 + (-2 - 3i)x^2 + x + (-2 - 3i)\right)$$

Now continue working with the reduced polynomial,

2 + 3i	1	–2 – 3i	1	–2 – 3i
		2 + 3i	0	2 + 3i
	1	0	1	0

Now continue working with the reduced plolynomial,

$$x^2 + 1 = 0 \Rightarrow x = \pm i$$

Therefore, the sum of the remaining zeros is:

$$(2 + 3i) + i + (-i)$$

$$= 2 + 3i$$

Ex 3 Verify that $P(x) = x^3 - i\,x^2 - x + i$ has i as a zero and that its conjugate $-i$ is not a zero. Explain why this does not contradict the Conjugate Zero Theorem.

Sol

Using the remainder theorem,

$$P(i) = i^3 - i\,i^2 - i + i$$

$$= -i + i - i + i = 0$$ So, i is a zero of $P(x)$

However, $P(-i) = (-i)^3 - i(-i)^2 - (-i) + i$

$$= -i + i + i + i = 2i \neq \mathbf{0}$$

So, $-i$ is not a zero of $P(x)$

This does not contradict the Conjugate Zero Theorem since NOT all the coefficients of $P(x)$ are real numbers.

Ex 4 Find the polynomial with lowest degree with **real coefficients** and zeros 2(multiplicity 3), – 5, i, 2+i.

Sol

The complex numbers $-i$, $2-i$ are also zeros ?

So the polynomial has at least 6 zeros, namely,

2(multiplicity 3), – 5, i, 2+i, $-i$, $2-i$

By the factor theorem, there must be six factors, namely,

$(x-2)^3$, $(x-(-5))$, $(x-i)$, $(x-(2+i))$, $(x-(-i))$, $(x-(2-i))$

A polynomial of lowest degree is the product of all factors

$$F(x) = a\,(x-2)^3(x-(-5))[(x-i)(x-(-i))][(x-(2+i))(x-(2-i))]$$

$$F(x) = a\,(x-2)^3(x+5))[x-i^2][(x-2)^2-i^2]$$

Use difference of two squares formula to put *P* in standard form

$$F(x) = a\,(x-2)^3(x+5))[x^2-i^2][(x-2)^2-i^2]$$

$$F(x) = a\,(x-2)^3(x+5))[x^2+1][x^2-4x+5]$$

Notice that there are other polynomials that serve the purpose, like,

$$F(x) = \mathbf{2}\,(x-2)^3(x+5))[x^2+1][x^2-4x+5]$$

$$F(x) = \mathbf{5}\,(x-2)^3(x+5))[x^2+1][x^2-4x+5]$$

Since we are given no information about F, we can choose any number for a , you may choose $a = 1$ *to* get

$$F(x) = (x-2)^3(x+5))[x^2+1][x^2-4x+5]$$

Ex 5 Find a polynomial $P(x)$ of degree 3 with **real coefficients** and zeros $1/2$, $1 - i$ where $P(4) = 105$.

Sol

The complex numbers $1 + i$ is also zeros ?

So the polynomial has at least three zeros, namely,

$$1/2,\ 1 - i,\ \ 1 + i$$

By the factor theorem, there must be three factors, namely,

$$(x - 1/2),\ (x - (1+i)),\ (x - (1 - i))$$

A polynomial of degree three is the product of all factors

$$P(x) = a\,(x - 1/2)\,(x - (1+i))\,(x - (1 - i))$$

$$P(x) = a\,(x - 1/2)\,((x - 1)^2 - i^2)$$

Use difference of two squares formula to put P in standard form

$$P(x) = a\,(x - 1/2)\,((x - 1)^2 - i^2)$$

Use perfect square formula to put P in standard from

$$P(x) = a\,(x - 1/2)\,(x^2 - 2x + 2)$$

$$P(x) = a\,(x^3 - 5/2\,x^2 + 3x - 1)$$

Perform multiplication and combine like terms

Since $P(4) = 105$, then:

$$105 = a\,((4)^3 - 5/2\,(4)^2 + 3\,(4) - 1)$$

which implies that $a = 3$

Thus, the polynomial is $P(x) = 3\,(x^3 - 5/2\,x^2 + 3x - 1)$

$$P(x) = 3x^3 - \frac{15}{2}x^2 + 9x - 3$$