3.3 ZEROS OF POLYNOMIAL FUNCTIONS

Objectives:

In this section, we will learn about:

- Multiple Zeros of a Polynomial Function.
- The Rational Zero Theorem.
- Upper and Lower Bounds for Real Zeros.
- Descartes’ Rule of Signs.
- Zeros of a Polynomial Function.

Multiple Zeros of a Polynomial Function

If a polynomial function $P(x)$ has $(x - c)$ as a factor exactly k times, then c is **a zero of multiplicity k** of the polynomial function $P(x)$.

Ex 1 If $P(x) = x^3(x - 2)^2(x + 3)$, then

- $x = 0$ is a zero of multiplicity 3
- $x = 2$ is a zero of multiplicity 2
- $x = -3$ is a zero of multiplicity 1 (simple zero)

Number of zeros of a polynomial Function

A polynomial function P of degree n has **at most n zeros**, where each zero of multiplicity k is counted k times.

Rational Zero Theorem

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ has **integer coefficients** $(a_n \neq 0)$, then **every rational zero** of P is of the form $\frac{p}{q}$ where

- p is a factor of the **constant** term a_0
- q is a factor of the **leading coefficient** a_n

Ex 2 Use the rational Zero Theorem to list all possible rational zeros of $P(x) = \underset{\substack{\uparrow \\ q \text{ factors of } 4}}{4}x^4 + x^3 - x^2 - 3x - \underset{\substack{\uparrow \\ p \text{ factors of } -12}}{12}$.

Sol

By the **Rational Zero Theorem**, the ***possible*** rational zeros of P are of the form

$$\frac{p}{q} = \frac{\text{factor of constant term}}{\text{factors of Leading coefficient}} = \frac{\text{factor of } -12}{\text{factors of } 4}$$

The factors of – 12 are p : $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

The factors of 4 are q : $\pm 1, \pm 2, \pm 4$

Thus, the ***possible rational*** zeros of P are: (simplify the fractions and remove duplicates)

$$\frac{p}{q}: \quad \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 2, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}, \pm 4, \pm 6, \pm 12$$

Upper and Lower Bounds for Real Zeros

- A real number b is called **an upper bound** of the zeros of the polynomial function P if **no zero is greater** than b.
- A real number b is called **lower bound** of the zeros of the polynomial function P if **no zero is less** than b.

Upper and Lower Bound Theorem:

Let $P(x)$ be a polynomial of degree $n \geq 1$ with **real coefficients**. If $P(x)$ is divided **synthetically** by $x - c$ and

(a) **Upper Bound: if *c* > *0*** and the **leading coefficient is positive** and all numbers in the **bottom row** of the synthetic division are **nonnegative**, then $P(x)$ has no zero greater than c.

(b) **Upper Bound: if *c* > *0*** and the **leading coefficient is negative** and all numbers in the **bottom row** of the synthetic division are **non positive**, then $P(x)$ has no zero greater than c.

(c) **Lower Bound: if *d* < *0*** and the numbers in the **bottom row** of the synthetic division **alternate in sign**(with 0 is considered positive or negative, as needed), then $P(x)$ has no zero less than d.

Ex 3 According to the Upper- and Lower- bound Theorem, what is the smallest positive integer that is an upper bound and the largest negative integer that is a lower bound of the real zeros of the following polynomials.

$$a)\ P(x) = x^3 + 3x^2 - 6x - 6$$

Sol

- To find the smallest positive integer that is an upper-bound, use synthetic division with the **positive integers** 1, 2 , 3, …, as test values

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -6 & -6 \\ & & 1 & 4 & \\ \hline & 1 & 4 & -2 & \end{array}$$

Criterion for Upper Bound:
***c > 0* & leading coefficient is positive**
all numbers in the bottom row of the synthetic division are **nonnegative**

1 is not an upper bound. Check the next positive integer.

$$\begin{array}{r|rrrr} 2 & 1 & 3 & -6 & -6 \\ & & 2 & 10 & 8 \\ \hline & 1 & 5 & 4 & 2 \end{array}$$

No negative numbers

Thus 2 is the **smallest** (?) positive integer that is an upper bound.

- To find the largest negative integer that is a lower- bound, use synthetic division with the negative integers –1, – 2 , – 3, …, as test values

$$\begin{array}{r|rrrr} -1 & 1 & 3 & -6 & -6 \\ & & -1 & & \\ \hline & 1 & 2 & & \end{array}$$

Criterion for Lower Bound:
***d < 0* &** the numbers in the **bottom row** of the synthetic division **alternate in sign**

–1 is not a lower bound. Check the next negative integer.

$$\begin{array}{r|rrrr} -2 & 1 & 3 & -6 & -6 \\ & & -2 & & \\ \hline & 1 & 1 & & \end{array}$$

–2 is not a lower bound. Check the next negative integer.

$$\begin{array}{r|rrrr} -3 & 1 & 3 & -6 & -6 \\ & & -3 & 0 & \\ \hline & 1 & 0 & -6 & \end{array}$$

–3 is not a lower bound. Check the next negative integer.

$$\begin{array}{r|rrrr} -4 & 1 & 3 & -6 & -6 \\ & & -4 & 4 & \\ \hline & 1 & -1 & -2 & \end{array}$$

–4 is not a lower bound. Check the next negative integer.

$$\begin{array}{r|rrrr} -5 & 1 & 3 & -6 & -6 \\ & & -5 & 10 & -20 \\ \hline & 1 & -2 & 4 & -26 \end{array}$$

Alternating signs

Thus –5 is the **largest (?)** negative integer that is a lower bound.

That is, $-5 <$ **all real zeros of P** < 2

Practice: Let $P(x)=-4x^4+12x^3+3x^2-12x+7$. Show that:

a) the smallest integer that is an upper bound is 4.

b) the largest integer that is a lower bound is –2.

Descartes' Rule of Signs

Let $P(x)$ be a polynomial function with **real coefficients** and with the terms arranged in order of **decreasing powers** of x

1. The number of ***positive real zeros*** of $P(x)$ either equals to the number of variations in sign occurring in the coefficients of $P(x)$, or is less than that number by an even integer.
2. The number of *negative real zeros* of $P(x)$ either equals to the number of variations in sign occurring in the coefficients of $P(-x)$, or is less than that number by an even integer.

A **variation in sign** means that two **consecutive**, nonzero coefficients have **opposite** signs.

Ex 4 Use Descartes' Rule of Signs to determine both the number of possible positive and the number of possible negative real zeros of each of the following polynomial functions.

a) $P(x) = 3x^4 - 12x^3 + 20x^2 + 7x - 16$

\+ to –　　　　　　+ to –

$$P(x) = 3x^4 - 12x^3 + 20x^2 + 7x - 16$$

– to +

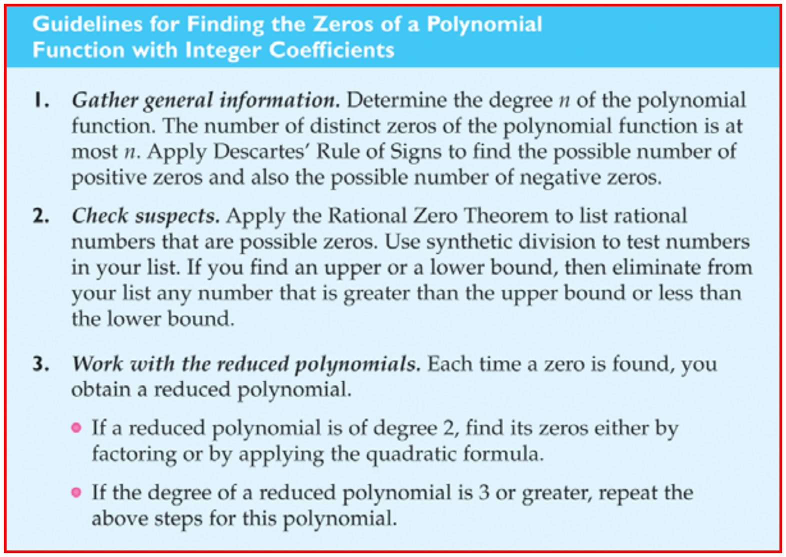
$P(x)$ has ***three*** **variations** in sign. Thus $P(x)$ has either three positive real zeros or one positive real zero.

$$P(-x) = 3(-x)^4 - 12(-x)^3 + 20(-x)^2 + 7(-x) - 16$$
$$= 3x^4 + 12x^3 + 20x^2 - 7x - 16$$

One change in sign

P(x) has one negative real zero.

Zeros of a Polynomial Function



Guidelines for Finding the Zeros of a Polynomial Function with Integer Coefficients

1. ***Gather general information.*** Determine the degree *n* of the polynomial function. The number of distinct zeros of the polynomial function is at most *n*. Apply Descartes' Rule of Signs to find the possible number of positive zeros and also the possible number of negative zeros.
2. ***Check suspects.*** Apply the Rational Zero Theorem to list rational numbers that are possible zeros. Use synthetic division to test numbers in your list. If you find an upper or a lower bound, then eliminate from your list any number that is greater than the upper bound or less than the lower bound.
3. ***Work with the reduced polynomials.*** Each time a zero is found, you obtain a reduced polynomial.
    - If a reduced polynomial is of degree 2, find its zeros either by factoring or by applying the quadratic formula.
    - If the degree of a reduced polynomial is 3 or greater, repeat the above steps for this polynomial.

Ex 5 Find the zeros of the polynomial $P(x) = x^3 + 3x^2 - 6x - 8$

Sol

Descartes' Rule of Signs: there are one positive and two or No negative real zeros

By the Rational Zero Theorem , the Possible Rational Zeros

$$\frac{p}{q} = \pm 1, \quad \pm 2, \quad \pm 4, \quad \pm 8$$

Use the synthetic division to test the positive possible(potential) rational zero, start by from the smallest

$$\begin{array}{r|rrrr} 1 & 1 & 3 & -6 & -8 \\ & & 1 & 4 & -2 \\ \hline & 1 & 4 & -2 & -10 \end{array}$$

1 is not a zero

$$\begin{array}{r|rrrr} 2 & 1 & 3 & -6 & -8 \\ & & 2 & 10 & 8 \\ \hline & 1 & 5 & 4 & 0 \end{array}$$

2 is a zero & all entries are nonnegative

We see that 2 is both a zero and an upper bound, so we don't need to check any further for positive zeros

$P(x)$ factors into $P(x) = (x-2)\left(x^2 + 5x + 4\right)$

Now, continue working with the reduced polynomial

$$(x-2)(x+4)(x+1) = 0$$

$$\Rightarrow x = 2, -4, \quad -1$$

The zeros of $P(x)$ are $2, -4$, and -1.

Ex 6 Find the *x*-intercepts of the graph of the polynomial function

$$P(x) = 3x^4 - 2x^3 - x^2 - 12x - 4$$

Sol

We need to solve the equation $3x^4 - 2x^3 - x^2 - 12x - 4 = 0$

Using the Rational Zero Theorem, the possible rational zeros are

$$\pm1, \pm2, \pm\frac{1}{3},, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm4$$

2	3	− 2	− 1	− 12	− 4
		6	8	14	4
	3	4	7	2	0

2 is a zero
& all entries are
nonnegative

So, *P* (*x*) factors into

$$P(x) = (x-2)\left(3x^3 + 4x^2 + 7x + 2\right)$$

$$P(x) = (x-2)\left(3x^3 + 4x^2 + 7x + 2\right)$$

Now we continue working with the quotient $3x^3 + 4x^2 + 7x + 2 = 0$

By the Descartes' Rule of Sign, $3x^3 + 4x^2 + 7x + 2$ has no positive zero.

So, its only possible rational zeros are – 1, –2, – 1 / 3 and – 2 / 3 .

– 1 / 3	3	4	7	2
		–1	–1	–2
	3	3	6	0

So, *P* (*x*) factors into

$$P(x) = (x-2)\left(x + \frac{1}{3}\right)\left(3x^2 + 3x + 6\right)$$

$$P(x) = (x-2)\left(x+\frac{1}{3}\right)\left(3x^2+3x+6\right)$$

Now we continue with the quotient $3x^2+3x+6=0$

The zeros of the quadratic factor are

$$x = \frac{-1 \pm \sqrt{1-8}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$$

So the *x*-intercepts of the graph of *P* are

$(2, 0), (-1 / 3, 0)$

Ex 7 Find the product of all zeros of the polynomial

Sol $P(x) = 2x^5 + 5x^4 - 8x^3 - 14x^2 + 6x + 9$

The possible rational zeros are $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3, \pm\frac{9}{2}$, and ± 9

Check the **positive candidates** first, beginning with the smallest

1 / 2	2	5	− 8	− 14	6	9
		1	3	− 5 / 2	− 33 / 8	15 / 16
	2	6	− 5	− 33 / 4	15 / 8	159 / 16

½ is not a zero

1	2	5	− 8	− 14	6	9
		2	7	− 1	−15	− 9
	2	7	−1	− 15	− 9	0

1 is a zero

and *P* (*x*) factors into

$$P(x) = (x-1)\left(2x^4+7x^3-x^2-15x-9\right)$$

$$P(x)=(x-1)\left(2x^4+7x^3-x^2-15x-9\right)$$

We continue working with the quotient(**reduced polynomial**)

We still have the same list of possible rational zeros ? except that 1 / 2 has been removed

1	2	7	– 1	– 15	– 9
		2	9	8	– 7
	2	9	8	– 7	– 16

1 is not a zero

3 / 2	2	7	– 1	– 15	– 9
		3	15	21	9
	2	10	14	6	0

3/ 2 is a zero
& all entries are
nonnegative

We see that 3 / 2 is both a zero and an upper bound, so we don't need to check any further for positive zeros

$$P(x)=(x-1)\left(x-\frac{3}{2}\right)\left(2x^3+10x^2+14x+6\right)$$

$$P(x)=(x-1)(2x-3)\left(x^3+5x^2+7x+3\right)$$

Factor 2 from the
last factor and
multiply into the
second factor

Now, we continue working with the reduced polynomial $x^3+5x^2+7x+3=0$

By the Descartes' Rule of Sign, x^3+5x^2+7x+3 has no positive zero.

So, its only possible rational zeros are – 1 and – 3.

– 1	1	5	7	3
		– 1	– 4	– 3
	1	4	3	0

– 1 is a zero

and *P* (*x*) factors into

$$P(x)=(x-1)(2x-3)(x+1)(x^2+4x+3)$$

Now continue working with the reduced polynomial which is quadratic and can be factored as

$$P(x) = (x-1)(2x-3)(x+1)(x+1)(x+3)$$

$$P(x) = (x-1)(2x-3)(x+1)^2(x+3)$$

This means that the zeros of P are 1, 3 / 2, – 1, and –3

and the product of the zeros is:

$$1 \cdot (3/2) \cdot (-1) \cdot (-3)$$

$$= 9/2$$