3.2 POLYNOMIAL FUNCTIONS OF HIGHER DEGREE

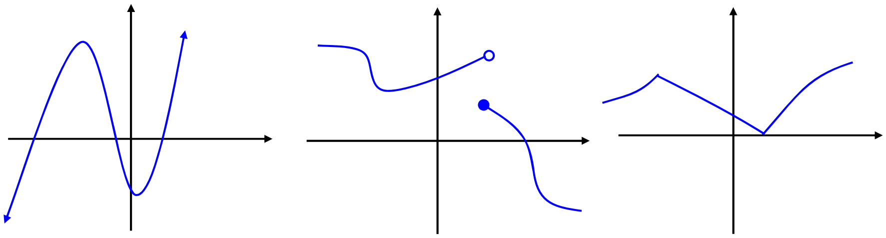
Objectives:

In this section, we will learn about:

- Far-Left and Far-Right Behavior.
- Maximum and Minimum Values.
- The Intermediate Value Theorem.
- Real Zeros, x intercepts and Factors of a Polynomial Function.
- Even and Odd Powers of (x – c) Theorem.
- A Procedure for Graphing Polynomial Functions.

All polynomial functions have graphs that are smooth continuous curves (منحنيات متصلة)

- A smooth curve: has rounded corners (no sharp corners).
- A continuous curve: no breaks, no holes, or no gaps.



continuous smooth polynomial	not continuous not polynomial	continuous not smooth not polynomial

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Far-Left and Far-Right Behavior

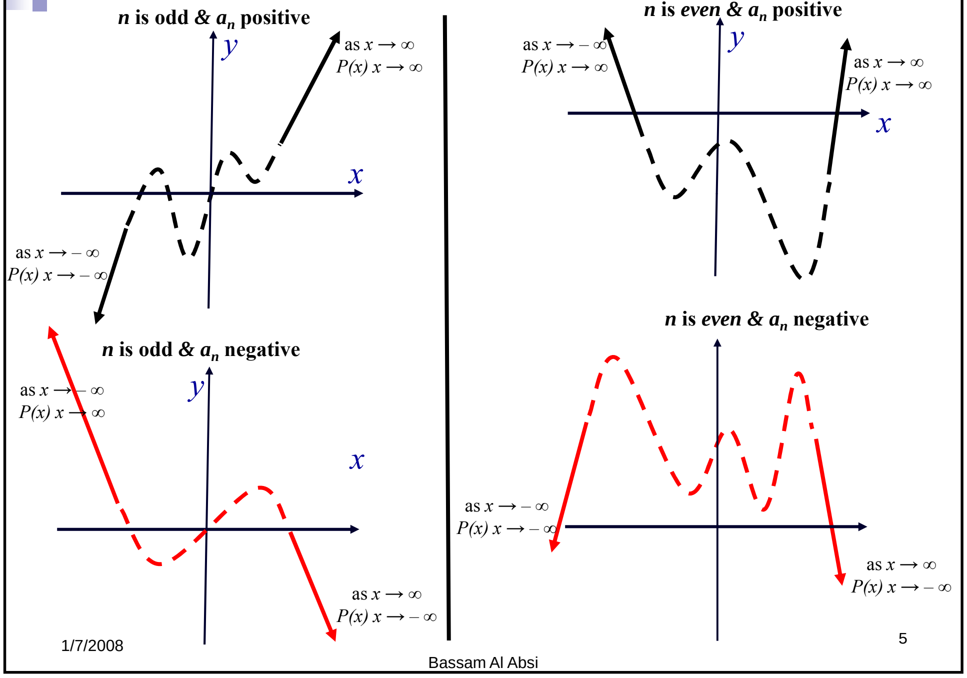
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ $(a_n \neq 0)$, then the **far-left and far right** behavior of the graph of $P(x)$ is determined by its **leading term** $a_n x^n$ as described in the below given table.

(Leading Coefficient Test)

	n is odd	**n is even**
$a_n > 0$	**Down** to far left and **Up** to far right.	**Up** to far left and **Up** to far right.
$a_n < 0$	**Up** to far left and **Down** to far right.	**Down** to far left and **Down** to far right.

Far - Right: as x grows positively without bound.

Far – Left : as x grows negatively without bound.



Ex 1 Examine the leading term to determine the far-left and far right behaviour of the graph of the following polynomial functions:

$$a)\ P(x) = 4x^3 + x^2 - 2x + 3$$

Sol

Since $a_n = 4$ is positive and $n = 3$ is odd, the graph of P goes **down** to its far **left** and **up** to its far **right**.

$$b)\ Q(x) = -\frac{1}{2}(x^6 - x^3 + 1)$$

Sol

Since $a_n = -\frac{1}{2}$ is negative and $n = 6$ is even, the graph of Q goes **down** to its far **left** and **down** to its far **right**.

$c)\quad R(x) = 2x^4 + x + 2$

Sol

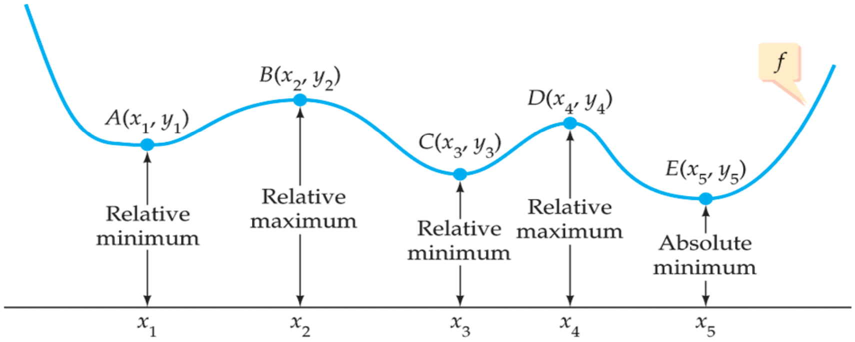
Since $a_n = 2$ is positive and $n = 4$ is even, the graph of R goes **up** to its far **left** and **up** to its far **right**.

$d)\ T(x) = 2(-3x+1)^3(x-2)^2$

Sol

Since $a_n = -54$ is negative and $n = 5$ is odd, the graph of T goes **up** to its far **left** and **down** to its far **right**.

Relative Maximum and Relative Minimum Values



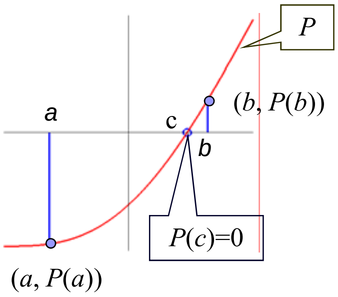
Definition:

A **turning point** of a graph of a function is a point at which the graph changes from **increasing** to **decreasing** or ***vice versa***.

A polynomial function of degree n has **at most $n - 1$** *turning points* and **at most *n zeros***.

The Zero Location Theorem (نظريةتحديد موقع الصفر)

Let $P(x)$ be a polynomial function and let be a and b be two distinct real numbers. If $P(a)$ and $P(b)$ have **opposite signs**, then there is **at least one real** number c between a and b such that $P(c) = 0$



Ex 2 Use the zero Location Theorem to verify that the polynomial

$P(x) = 2x^3 - 21x^2 - 2x + 25$ has a zero between 1 and 2.

Sol

1	2	–21	–2	25
		2	–19	–21
	2	–19	–21	4

$P(1)$ *is* positive

2	2	–21	–2	25
		4	–34	–72
	2	–17	–36	–47

$P(2)$ *is* negative

$P(1)$ is positive; $P(2)$ is negative. Therefore, by the Zero Location Theorem, $P(x)$ has a zero between 1 and 2.

Definition: A real number c is a **zero** of a function $y = f(x)$ if and only if $f(c) = 0$.

Real Zeros of Polynomial Functions

If $y = f(x)$ is a polynomial function and c is a real number then the following statements are equivalent (متكافئة).

1. c is a zero of f.
2. c is a solution of the polynomial equation $f(x) = 0$.
3. $x - c$ is a factor of the polynomial $f(x)$.
4. $(c, 0)$ is an x-intercept of the graph of $y = f(x)$.

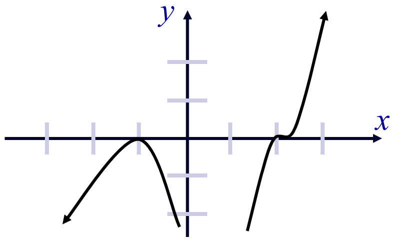
Repeated Zeros :

If k is the largest integer for which $\mathbf{(x - a)^k}$ is a **factor** of $f(x)$ and $k > 1$, then a is a **repeated zero** of **multiplicity** k.

1. If k is ***odd*** the graph of $f(x)$ **crosses the *x*-axis** at $(a, 0)$.
2. If k is ***even*** the graph of $f(x)$ **touches, but does not cross** through, the x-axis at $(a, 0)$.

Ex 3 Determine the multiplicity of the zeros of $f(x) = (x - 2)^3(x + 1)^4$.

Zero	Multiplicity	Behavior
2	3 odd	crosses x-axis at (2, 0)
–1	4 even	touches x-axis at (–1, 0)



A Procedure for Graphing Polynomial Functions:

1. Determine the far-left and far-right behavior.
2. Find the y-intercept.
3. Find the x-intercept(s) and determine the behavior of the graph near the x-intercept(s).
4. Use the number line test to determine the regions in which the graph lies above or below the x-axis.
5. Find additional point on the graph.
6. Check for symmetry.
   a. The graph of an even function is symmetric with respect to the y-axis.
   b. The graph of an odd function is symmetric with respect to the origin.

6. Use all the above information to sketch the graph

Ex 4 Sketch the graph of $P(x) = 4x^2 - x^4$.

Sol

1. Write the $P(x)$ in standard form: $f(x) = -x^4 + 4x^2$

The leading coefficient is ***negative*** and the degree is ***even***.

The graph goes down to its far left and right.

2. Find the y-intercept:(0, 0)

3. Find the x – intercept(s):

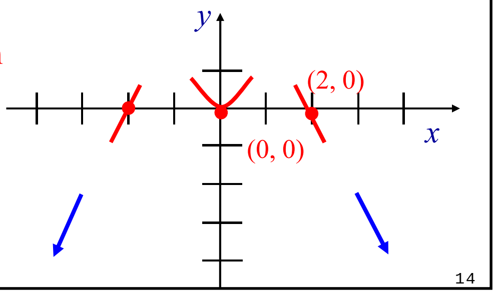
$-x^4 + 4x^2 = 0 \qquad -x^2(x^2 - 4) = 0$

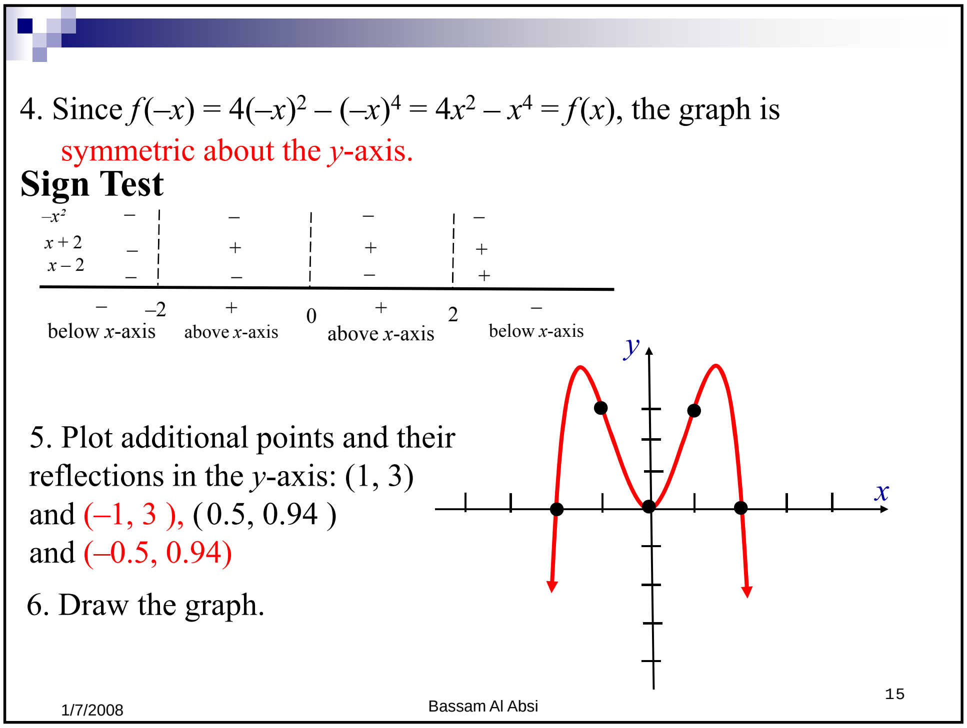
$-x^2(x + 2)(x - 2) = 0$

***x*-intercepts:**

(–2, 0), (2, 0) crosses through

(0, 0) Touches x-axis only





4. Since $f(-x) = 4(-x)^2 - (-x)^4 = 4x^2 - x^4 = f(x)$, the graph is symmetric about the y-axis.

Sign Test

$-x^2$	−		−		−		−
$x + 2$	−		+		+		+
$x - 2$	−		−		−		+
	−	−2	+	0	+	2	−
	below x-axis		above x-axis		above x-axis		below x-axis

5. Plot additional points and their reflections in the y-axis: (1, 3) and (–1, 3), (0.5, 0.94) and (–0.5, 0.94)

6. Draw the graph.