نظرية الباقي

3.1 THE REMAINDER THEOREM
AND
THE FACTOR THEOREM

نظرية العوامل

Objectives:

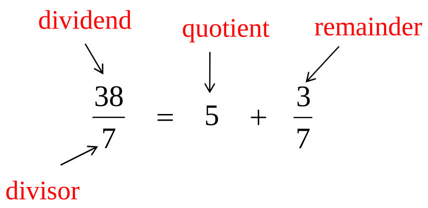
In this section, we will learn about:

- Division of Polynomials.
- Synthetic Division.
- The Remainder Theorem.
- The Factor Theorem.
- The Reduced Polynomials.

Dividing Polynomials

Dividing polynomials is much like the familiar process of dividing numbers.

For example, when we divide 38 by 7, we write:



dividend quotient remainder

$$\frac{38}{7} = 5 + \frac{3}{7}$$

divisor

Notations and Rules on Polynomials Division:

1)
- $P(x)$: Dividend المقسوم
- $D(x)$: Divisor المقسوم عليه
- $Q(x)$: Quotient ناتج القسمة
- $R(x)$: Remainder الباقي

$$\boxed{\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$$

2) The degree of $R(x) <$ the degree of $D(x)$

3) Before dividing polynomials, make sure that each polynomial is written in **descending order**.

4) **Insert** a 0 a **missing term**.

Ex 1 Divide $x^2 + 3x - 2$ by $x + 2$ and check the answer.

Sol

$$
\begin{array}{rl}
 & x + 1 \\ \cline{2-2}
x+2 \,\big) & x^2 + 3x - 2 \\
 & \underline{x^2 + 2x} \quad \downarrow \\
 & \qquad\quad x - 2 \\
 & \qquad\quad \underline{x + 2} \\
 & \qquad\qquad \underset{\underset{\text{remainder}}{\uparrow}}{-4}
\end{array}
$$

Thus $\dfrac{x^2 + 3x - 2}{x + 2} = x + 1 + \dfrac{-4}{x + 2}$

Check: $\underset{\text{quotient}}{(x + 1)}\underset{\text{divisor}}{(x + 2)} + \underset{\text{remainder}}{(-4)} \; \underset{\text{dividend}}{= x^2 + 3x - 2}$

Ex 2 Perform the operation $\frac{3x^3-7x^2+2x^4-10}{x^2-2x-5}$

Sol

$$\begin{array}{r l}
 & 2x^2 + 7x + 17 \\ \cline{2-2}
x^2-2x-5 \,\big) & 2x^4+3x^3-7x^2+0\cdot x - 10 \\
 & 2x^4 - 4x^3 - 10x^2 \quad \downarrow \quad \downarrow \\ \cline{2-2}
 & \qquad 7x^3 + 3x^2 + 0x \\
 & \qquad 7x^3 - 14x^2 - 35x \\ \cline{2-2}
 & \qquad\qquad 17x^2 + 35x - 10 \\
 & \qquad\qquad 17x^2 - 34x - 85 \\ \cline{2-2}
 & \qquad\qquad\qquad 69x + 75
\end{array}$$

Thus $\frac{3x^3-7x^2+2x^4-10}{x^2-2x-5} = 2x^2+7x+17 + \frac{69x+75}{x^2-2x-5}$

Synthetic Division is a **shortcut** method of dividing certain polynomials. This method can be used only when the **divisor** is of the form $x - c$ where the **coefficient of x is 1**. It uses the coefficients of each term in the dividend.

Ex 3 Divide $2x^2 - 4x - 1$ by $x + 3$ using synthetic division.

Sol

Since the divisor is $x + 3$, $c = -3$.

value of c

coefficients of the dividend

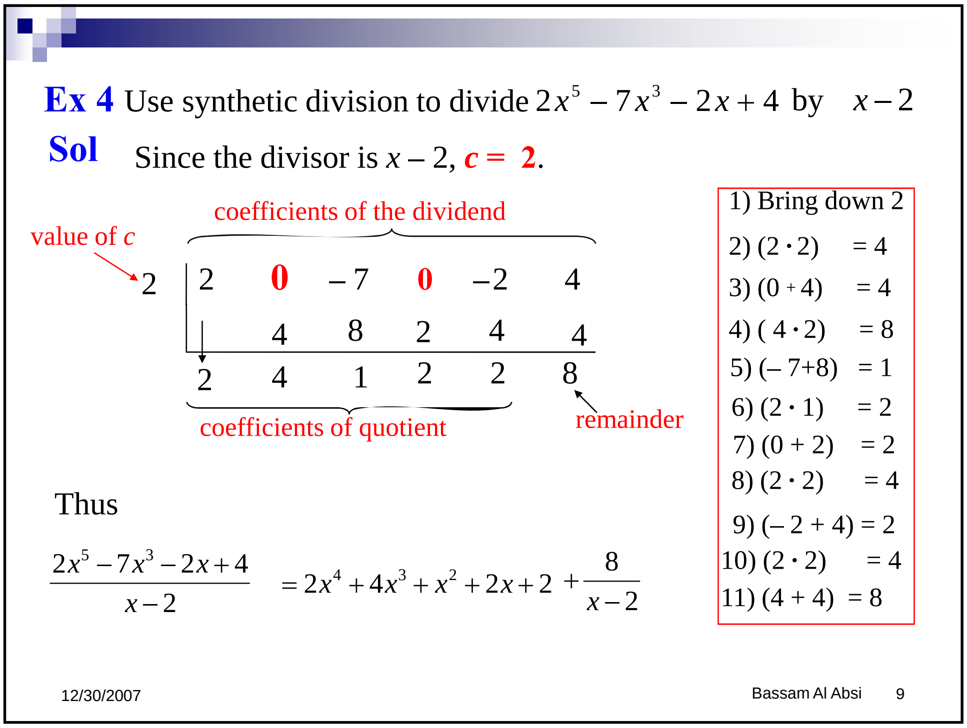
$$\begin{array}{r|rrr}
-3 & 2 & -4 & -1 \\
 & \downarrow & -6 & 30 \\ \hline
 & 2 & -10 & 29
\end{array}$$

coefficients of quotient

remainder

1) Bring down 2
2) $(2 \cdot -3) = -6$
3) $(-4 + (-6)) = -10$
4) $(-3 \cdot -10) = 30$
5) $(-1 + 30) = 29$

Answer: $2x - 10 + \frac{29}{x+3}$



Ex 4 Use synthetic division to divide $2x^5-7x^3-2x+4$ by $x-2$

Sol Since the divisor is $x-2$, $c=2$.

value of c

coefficients of the dividend

$$\begin{array}{r|rrrrrr} 2 & 2 & 0 & -7 & 0 & -2 & 4\\ & \downarrow & 4 & 8 & 2 & 4 & 4 \\ \hline & 2 & 4 & 1 & 2 & 2 & 8\end{array}$$

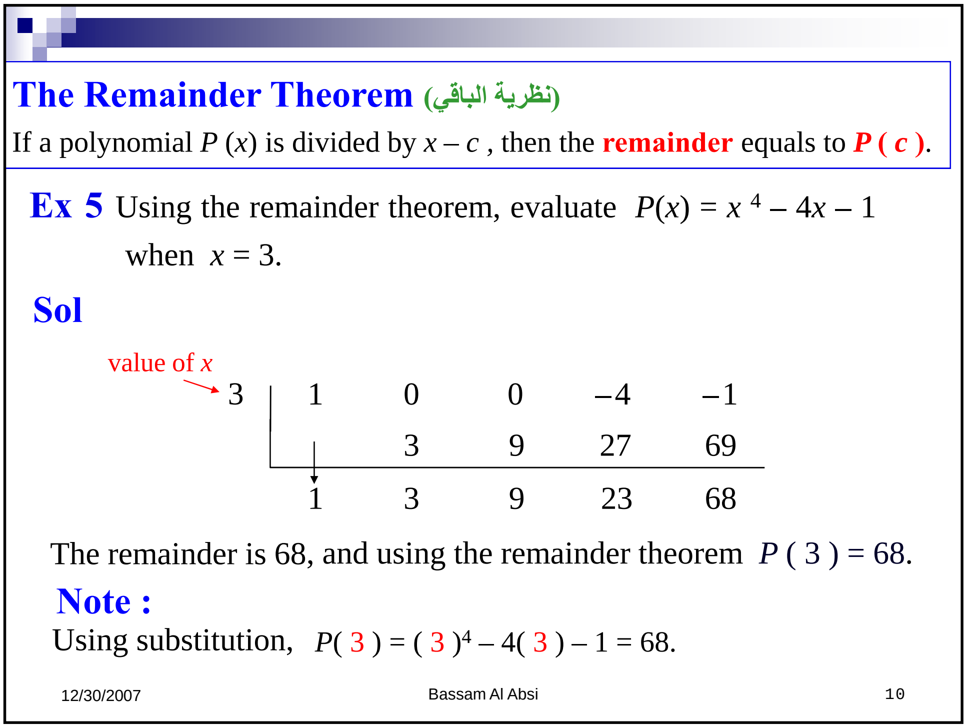
coefficients of quotient

remainder

1) Bring down 2
2) (2 · 2) = 4
3) (0 + 4) = 4
4) (4 · 2) = 8
5) (– 7+8) = 1
6) (2 · 1) = 2
7) (0 + 2) = 2
8) (2 · 2) = 4
9) (– 2 + 4) = 2
10) (2 · 2) = 4
11) (4 + 4) = 8

Thus

$$\frac{2x^5-7x^3-2x+4}{x-2} = 2x^4+4x^3+x^2+2x+2+\frac{8}{x-2}$$



The Remainder Theorem (نظرية الباقي)

If a polynomial $P(x)$ is divided by $x-c$, then the **remainder** equals to $P(c)$.

Ex 5 Using the remainder theorem, evaluate $P(x) = x^4 - 4x - 1$ when $x = 3$.

Sol

value of x

$$\begin{array}{r|rrrrr} 3 & 1 & 0 & 0 & -4 & -1\\ & \downarrow & 3 & 9 & 27 & 69 \\ \hline & 1 & 3 & 9 & 23 & 68\end{array}$$

The remainder is 68, and using the remainder theorem $P(3) = 68$.

Note :

Using substitution, $P(3) = (3)^4 - 4(3) - 1 = 68$.

Ex 6 If $P(x) = 211x^{4} - 212x^{3} + 212x^{2} + 210x - 3$, then find $P(1/211)$.

Sol

$P(1/211)$ can be found by
- → Substitution (**Try**) a bit complicated process
- → Synthetic Division (**Very simple**)

Lets try the synthetic division

$$\begin{array}{r|rrrrr} 1/211 & 211 & -212 & 212 & 210 & -3 \\ & \downarrow & 1 & -1 & 1 & 1 \\ \hline & 211 & -211 & 211 & 211 & -2 \end{array}$$

remainder

Thus, $P(1/211) = -2$

Ex 7 Find the remainder when $P(x) = x^{103} + x^{102} + x^{101} + x^{100}$ is divided by $x + i$.

Sol

The remainder can be found by
- → Substitution (**Very simple**)
- → Synthetic Division (**Try**) very complicated process

Lets try the substitution

The remainder = $P(-i)$

$$= (-i)^{103} + (-i)^{102} + (-i)^{101} + (-i)^{100}$$

$$= -i^{3} + i^{2} - i + i^{0}$$

$$= i - 1 - i + 1$$

$$= 0$$

The Factor Theorem (نظرية العوامل)

A polynomial $P(x)$ has a **factor** $(x - c)$ if and only if $\boldsymbol{P(c) = 0}$.

Ex 8 Show that $x + 2$ is a factors of $P(x) = 2x^3 + x^2 - 5x + 2$.

Sol

$$\begin{array}{r|rrrr} -2 & 2 & 1 & -5 & 2 \\ & \downarrow & -4 & 6 & -2 \\ \hline & 2 & -3 & 1 & 0 \end{array}$$

The remainder of 0 indicates that $x + 2$ is a factor $P(x)$.

Try:

In the above example, show that $(x + 1)$ is another factor of $P(x)$.

Result: A real number c is a **zero** of $P(x)$ if and only if $P(c) = 0$.

Real Zeros of Polynomial Functions

If $P(x)$ is a polynomial and c is a real number then the following statements are equivalent.

1. $x = c$ is a ***zero*** of P.
2. $x = c$ is a ***solution*** of the polynomial equation $P(x) = 0$.
3. $(x - c)$ is a factor of the polynomial $P(x)$.
4. $(c, 0)$ is an x-intercept of the graph of $P(x)$.

Notation: If $\boldsymbol{P(x) = (x - c)\,Q(x)}$, where $\boldsymbol{Q(x)}$ is a polynomial of degree < degree of $\boldsymbol{P(x)}$ by 1 , then $\boldsymbol{Q(x)}$ is called the ***reduced polynomial.***

Ex 9 Verify that $(x + 1)$ is a factor of $P(x) = x^4 + 5x^3 + 3x^2 - 5x - 4$, and write $P(x)$ as the product of $(x + 1)$ and the reduced polynomial $Q(x)$.

Sol

–1	1	5	3	–5	–4
		–1	–4	1	4
	1	4	–1	– 4	**0**

A remainder of 0 implies that $x + 1$ is a factor of $P(x)$.

$$P(x) = (x+1)\underbrace{(x^3 + 4x^2 - x - 4)}_{\text{reduced polynomial}}$$

Ex 10 Prove that for any positive odd integer n, $x + 1$ is a factor of $P(x) = x^n + 1$.

Sol

Using the remainder theorem, $x + 1$ is a factor of $P(x) = x^n + 1$ if $P(-1) = 0$?

The remainder $P(-1) = (-1)^n + 1$
$= -1 + 1$
$= 0$

$\boxed{\text{, for any odd integer, } (-1)^n = -1}$

Thus, $x + 1$ is a factor of $P(x) = x^n + 1$

Ex 11 If $x - i$ is a factor of the polynomial

$P(x) = 7x^{171} - 8x^{172} - 9x^{173} + kx^{174}$, then find the value of k.

Sol

Since $x - i$ is a factor, then using the remainder theorem, $P(i) = 0$

, that is,

$$7i^{171} - 8i^{172} - 9i^{173} + ki^{174} = 0$$

$$7i^{3} - 8i^{0} - 9i + ki^{2} = 0$$

$$-7i - 8 - 9i - k = 0$$

Evaluate powers of i and combine like terms

Thus $k = -8 - 16i$