2.6 ALGEBRA OF FUNCTIONS

Objectives:

In this section, we will learn about:

- Operations on Functions.
- The Difference Quotient.
- Composition of Functions.

Operations on Functions:

Let f and g be two functions with domains D_f *and* D_g, then we define four new functions, namely, sum, difference, product and quotient of f and g as follows:

$$(f+g)(x) = f(x) + g(x) \qquad D_{f+g} = D_f \cap D_g$$

$$(f-g)(x) = f(x) - g(x) \qquad D_{f-g} = D_f \cap D_g$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \qquad D_{fg} = D_f \cap D_g$$

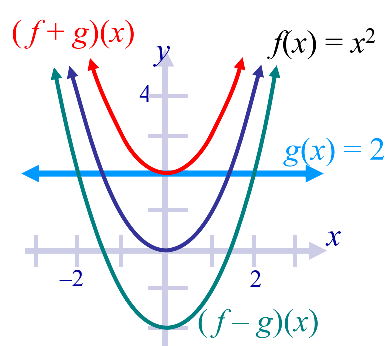
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0 \qquad D_{\frac{f}{g}} = D_f \cap D_g \text{ except } \{x \mid g(x) = 0\}$$

Ex 1 Given $f(x) = x^2$ and $g(x) = 2$, find $(f+g)(x)$ and $(f-g)(x)$.

Sol

$$(f+g)(x) = f(x) + g(x)$$
$$= x^2 + 2$$

$$(f-g)(x) = f(x) - g(x)$$
$$= x^2 - 2$$



Ex 2 Given $f(x) = 10 - 3x^3$ and $g(x) = 2x^2 + 1$, find $(f + g)(2)$ and $(f - g)(-1)$.

Sol

$$(f + g)(x) = f(x) + g(x) = (10 - 3x^3) + (2x^2 + 1)$$
$$= -3x^3 + 2x^2 + 11$$

$$(f + g)(2) = -3(2)^3 + 2(2)^2 + 11$$
$$= -5$$

$$(f - g)(x) = f(x) - g(x) = (10 - 3x^3) - (2x^2 + 1)$$
$$= -3x^3 - 2x^2 + 9$$

$$(f - g)(-1) = -3(-1)^3 - 2(-1)^2 + 9$$
$$= 10$$

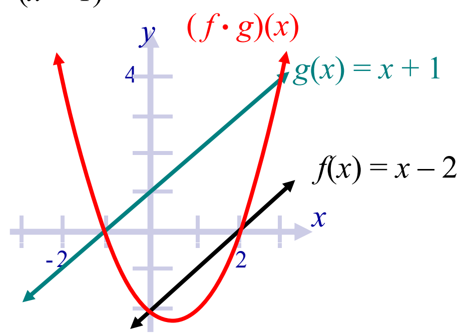
Ex 3 Given $f(x) = x - 2$ and $g(x) = x + 1$, find $(f \cdot g)(x)$ and $(f \cdot g)(3)$ and $(f \cdot g)(-1)$.

Sol

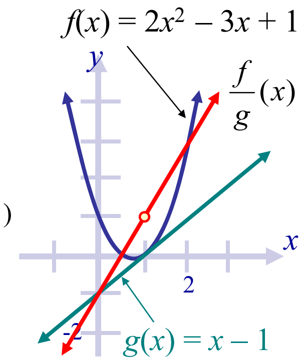
$$(f \cdot g)(x) = f(x) \cdot g(x) = (x - 2) \cdot (x + 1)$$
$$= x^2 - x - 2$$

$$(f \cdot g)(3) = ((3) - 2) \cdot ((3) + 1)$$
$$= (3)^2 - (3) - 2$$
$$= 4$$

$$(f \cdot g)(-1) = (-1)^2 - (-1) - 2$$
$$= 0$$



Ex 4 Given $f(x) = 2x^2 - 3x + 1$ and $g(x) = x - 1$, find $\frac{f}{g}(1)$ and $\frac{f}{g}(2)$.



Sol

$$\frac{f}{g}(1) = \frac{f(1)}{g(1)} = \frac{2(1)^2 - 3(1) + 1}{(1) - 1} = \frac{0}{0}$$

← This is not defined, so $\frac{f}{g}(1)$ cannot be determined.

$$\frac{f}{g}(2) = \frac{2(2)^2 - 3(2) + 1}{(2) - 1} = \frac{3}{1} = 3$$

Ex 5 Find the domain of $(f + g)$ and $\left(\frac{f}{g}\right)$ where

$$f(x) = \sqrt{16 - x^2} \quad \text{and} \quad g(x) = x^2 + 4x$$

Sol

Since $g(x)$ is a polynomial, then $D_g = (-\infty, \infty)$

For the domain of $f(x)$, exclude all real numbers that causes square root of negative numbers , that is to say,

$$16 - x^2 \ge 0$$

One way to solve this nonlinear inequality is to use the critical value method, or alternatively, we can do the following:

$$x^2 \le 16$$

Common Mistake: $x \le \pm 4$

$$\Rightarrow |x| \le 4 \quad \Rightarrow -4 \le x \le 4 \quad \Rightarrow D_f = [-4, 4]$$

$So\ we\ have\ \ f(x)=\sqrt{16-x^2}$ and $g(x)=x^2+4x$

$$D_g=(-\infty,\infty) \quad and \quad D_f=[-4,4]$$

Thus, $D_{f+g}=D_f\cap D_g=(-\infty,\infty)\cap[-4,4] \quad =[-4,4]$

$$D_{\frac{f}{g}}=D_f\cap D_g\ except\{\,x|g(x)=0\}$$

$$g(x)=0$$
$$\Rightarrow x^2+4x=0$$
$$if\ \ x=0,-4$$

$$D_{\frac{f}{g}}=[-4,4]except\{\,0,-4\}$$

$$D_{\frac{f}{g}}=(-4,0)\cup(0,4]$$

Difference Quotient:

The expression $\dfrac{f\,(\,x+h\,)-f\,(\,x\,)}{h}$ is called the **difference quotient** of f.

It enables us to study the manner in which a function changes in value as the independent variable changes.(Important for Calculus)

Ex 6 Find the difference quotient of $f(x)=2x-x^2$

Sol

$$\frac{f(x+h)-f(x)}{h}=\frac{\left[2(x+h)-(x+h)^2\right]-\left[2x-x^2\right]}{h}$$, substitute

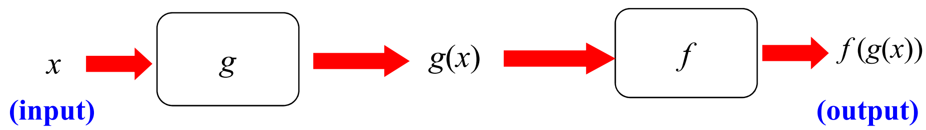
$$=\frac{2x+2h-x^2-2hx-h^2-2x+x^2}{h}$$, expand and simplify

$$=\frac{2h-2hx-h^2}{h}=\frac{h(2-2x-h)}{h}$$, takeout common factor

$$=2-2x-h$$

Composition of Functions : (تركيب الدوال)

A function can be evaluated at the value of another function.

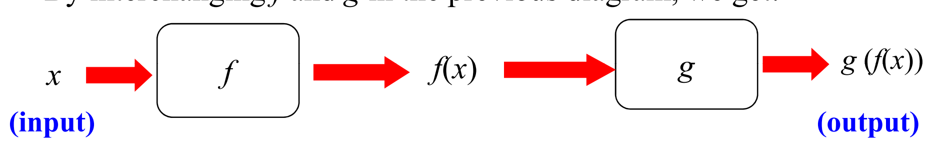


This operation is called the **composite** of the f and g.
it is denoted by $(f \circ g)$, and its value at x is given by:

$$(f \circ g)(x) = f\left[g(x)\right], \quad \text{, where } g(x) \in D_f \text{ for all } x \in D_g$$

read as "f circle g of x" or "f of g of x"

By interchanging f and g in the previous diagram, we get:



This operation is called the **composite** of the g and f.
it is denoted by $(g \circ f)$, and its value at x is given by:

$$(g \circ f)(x) = g\left[f(x)\right], \quad \text{, where } f(x) \in D_g \text{ for all } x \in D_f$$

read as "g circle f of x" or "g of f of x"

Ex 7 Given $f(x) = 2x + 5$ and $g(x) = x^2 + 1$, find $(f \circ g)(x)$ and $f[g(-2)]$.

Sol

$$(f \circ g)(x) = f[g(x)] = f[x^2 + 1] =$$

$$= 2(x^2 + 1) + 5 = 2x^2 + 7$$

$$f[g(-2)] = 2(-2)^2 + 7 = 15$$

Ex 8 If $f(x) = x^2 - 3,\ x < 0$ and $g(x) = \sqrt{x+3}$, find

a) $(f \circ g)(x)$

b) $(g \circ f)(x)$

c) $(f \circ f)(x)$

d) $(f \circ g \circ f)(x)$

Sol

$$a)\ (f \circ g)(x) = f[g(x)]$$
$$= f\left(\sqrt{x+3}\right)$$
$$= \left(\sqrt{x+3}\right)^2 - 3$$
$$= x$$

Result: $f \circ g \neq g \circ f$

$$b)\ (g \circ f)(x) = g[f(x)]$$
$$= g(x^2 - 3)$$
$$= \sqrt{x^2 - 3 + 3}$$
$$= \sqrt{x^2}$$
$$= |x| \quad ?$$
$$= -x$$

$$c)\ (f \circ f)(x) = f[f(x)]$$
$$= f(x^2 - 3)$$
$$= (x^2 - 3)^2 - 3$$
$$= x^4 - 6x^2 + 6$$

$$d)\ (f \circ g \circ f)(x) = (f \circ g)[f(x)]$$
$$= (f \circ g)(x^2 - 3)$$
$$= f[g(x^2 - 3)]$$
$$= f(-x)$$
$$= x^2 - 3$$

Ex 9 If $f(x)=2+3x-x^2$ and $g(x)=2x-1$, find

$a)\ (fg)(x)$ $b)\ (f\circ g)(x)$

Sol

a) $(fg)(x)$, *read as "f times g at x"*

$$=f(x)\bullet g(x)$$, product of functions f and g

$$=\left(2+3x-x^2\right)\left(2x-1\right)$$

$$=-2x^3+7x^2+x-2$$

b) $(f\circ g)(x)$, *read as "composite f and g"*

$$=f(g(x))$$
$$=f(2x-1)$$
$$=2+3\left(2x-1\right)-\left(2x-1\right)^2=-4x^2+10x$$

Ex 10 If $f(x)=2x+5, g(x)=5x+2k$ and $(f\circ g)(2)=10$, then find k.

Sol

$$(fog)(2)=10$$

$$f\left(g(2)\right)=10$$

$$f\left(5x+2k\right)=10$$

$$2\left(5\cdot 2+2k\right)+5=10$$

$$20+4k+5=10$$

$$k=\frac{-15}{4}$$

Ex 11 If $f(x)=3x+2$ and $(f\circ h)(x)=5x-1$, then find $h(x)$.

Sol

$$(f\circ h)(x)=5x-1$$

$$\downarrow$$

$$f(h(x)) = 5x-1$$, notice that $h(x)$ is not known, plug it in f as is

$$3\left(h\left(x\right)\right)+2=5x-1$$, solve for $h(x)$

$$3\left(h\left(x\right)\right)=5x-3$$

$$\Rightarrow h\left(x\right)=\frac{5}{3}x-1$$

Ex 12 Given $f(x)=x^2+x$ and $g(x)=x+k$, find all values of k such that $(fog)(3)=(gof)(3)$

Sol

$$(fog)(3)=(gof)(3)$$

$$f\left(g\left(3\right)\right) = g\left(f\left(3\right)\right)$$

$$f\left(3+k\right) = g\left(3^2+3\right)$$

$$\left(3+k\right)^2+\left(3+k\right) = 12+k$$

Write this quadratic equation in standard form and solve

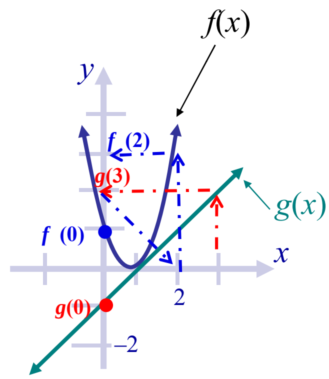
$$k^2+6k=0$$

$$k^2+6k=0$$

Thus, $k=0,-6$

Ex 13 Use the adjacent graph to evaluate: $a)\ \dfrac{f}{g}(0)$

$b)\ (fog)(-1)$



Sol

$$a)\ \frac{f}{g}(0)=\frac{f(0)}{g(0)}=\frac{1}{-1}=-1$$

$$b)\ (fog)(3) = f\left(g(3)\right) = f(2) = 3$$

Ex 14 Let $f(x)=\dfrac{1}{x}$ and $g(x)\sqrt{3-x}$, find domain of gof.

Sol

The domain of $g\circ f$ is the set of all real numbers x such that

Remember: $gof(x)=g\left(f(x)\right)$ $for\ all\ x\in D_f\ and\ f(x)\in D_g$.

$$x\neq 0\quad and\ 3-f(x)\geq 0$$

$$x\neq 0\quad and\ 3-\frac{1}{x}\geq 0$$

$$x\neq 0\quad and\ \frac{3x-1}{x}\geq 0$$

Use the critical value method to solve this inequality

The domain of gof is $(-\infty,0)\cup\left[\dfrac{1}{3},\infty\right)$