2.5 PROPERTIES OF GRAPHS

Objectives:

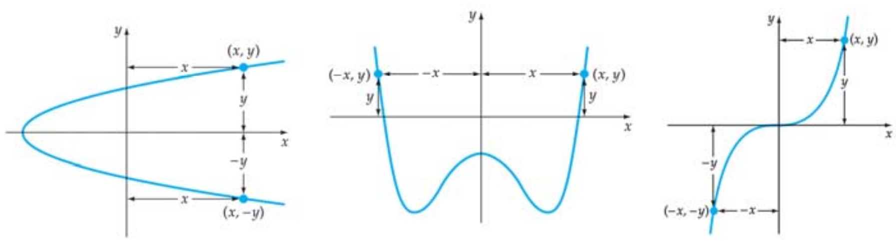
In this section, we will learn about, the following properties of graphs:

- Symmetry.
- Even and Odd Functions.
- Reflection.
- Vertical Translation(shift):
  - Up.
  - Down.
- Horizontal Translation(shift):
  - Right.
  - Left.

I) Symmetry (التناظر)

The Graph of $y = f(x)$ is:

- **Symmetric with Respect to the *x*-axis:** If (x, y) is on the graph, then $(x, -y)$ is also in the graph.
- **Symmetric with Respect to the *y*-axis:** If (x, y) is on the graph, then $(-x, y)$ is also in the graph.
- **Symmetric with Respect to the origin:** If (x, y) is on the graph, then $(-x, -y)$ is also in the graph.



Ex 1 Determine whether the graph of each of the following equations is symmetric with respect to the (I) *x*-axis,

(II) y-axis (III) origin.

$a)\quad y = x^2 + 1$

Sol

(I) Symmetry w.r.t. the x-axis: Let (x, y) be a point on the graph,

Question: Is the point $(x, -y)$ also in the graph?

Replace $y \to -y \quad \Rightarrow (-y) = x^2 + 1$

$\Rightarrow\ y = -x^2 - 1$

, notice that the last equation is ***different from*** the ***original*** equation, this means that the point $(x, -y)$ is ***not on the graph.***

Thus, the graph of the equation is ***not symmetric*** w.r.t. the x-axis.

(II) Symmetry w.r.t. the y-axis: Let (x, y) be a point on the graph,
Question: Is the point $(-x, y)$ also in the graph?

Replace $x \rightarrow -x \quad \Rightarrow \quad y = (-x)^2 + 1$

$\Rightarrow \quad y = x^2 + 1$

, notice that the last equation is the ***same as*** the ***original*** equation, this means that the point $(-x, y)$ is **on *the graph.***

Thus, the graph of the equation is ***symmetric*** w.r.t. the y-axis.

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(III) Symmetry w.r.t. the origin: Let (x, y) be a point on the graph,
Question: Is the point $(-x, -y)$ also in the graph?

Replace $x \rightarrow -x$ ***and*** $y \rightarrow -y \quad \Rightarrow \quad (-y) = (-x)^2 + 1$

$\Rightarrow \quad y = -x^2 - 1$

, notice that the last equation is ***different from*** the ***original*** equation, this means that the point $(-x, -y)$ is ***not on the graph.***

Thus, the graph of the equation is **not *symmetric*** w.r.t. the origin.

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$b) \quad y = \dfrac{x^2+1}{x^3}$

Sol

(I) Symmetry w.r.t. the x-axis: Let (x, y) be a point on the graph,
Question: Is the point $(x, -y)$ also in the graph?

Replace $y \to -y \quad \Rightarrow \quad -y = \dfrac{x^2+1}{x^3}$

$\Rightarrow \quad y = -\dfrac{x^2+1}{x^3}$

, notice that the last equation is ***different from*** the ***original*** equation, this means that the point $(x, -y)$ is ***not on the graph.***

Thus, the graph of the equation is ***not symmetric*** w.r.t. the x-axis.

(II) Symmetry w.r.t. the y-axis: Let (x, y) be a point on the graph,
Question: Is the point $(-x, y)$ also in the graph?

Replace $x \to -x \quad \Rightarrow \quad y = \dfrac{(-x)^2+1}{(-x)^3}$

$\Rightarrow \quad y = -\dfrac{x^2+1}{x^3}$

, notice that the last equation is ***different from*** the ***original*** equation, this means that the point $(-x, y)$ is not **on *the graph.***

Thus, the graph of the equation is ***not symmetric*** w.r.t. the y-axis.

(III) Symmetry w.r.t. the origin: Let (x, y) be a point on the graph,
Question: Is the point $(-x, -y)$ also in the graph?

Replace $x \to -x$ ***and*** $y \to -y \Rightarrow (-y) = \frac{(-x)^2 + 1}{(-x)^3}$

$\Rightarrow \quad y = \frac{x^2 + 1}{x^3}$

, notice that the last equation is the ***same as*** the ***original*** equation, this means that the point $(-x, -y)$ is ***on the graph.***

Thus, the graph of the equation is ***symmetric*** w.r.t. the origin.

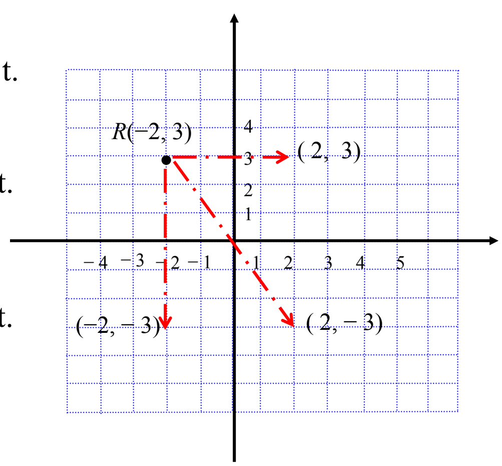
Ex 2 Find the image of the point $R(-2, 3)$ with respect to the x-axis, y-axis, and the origin

Sol

The image *of* $R(-2, 3)$ w.r.t. the x-axis is $(-2, -3)$

The image *of* $R(-2, 3)$ w.r.t. the y-axis is $(2, 3)$

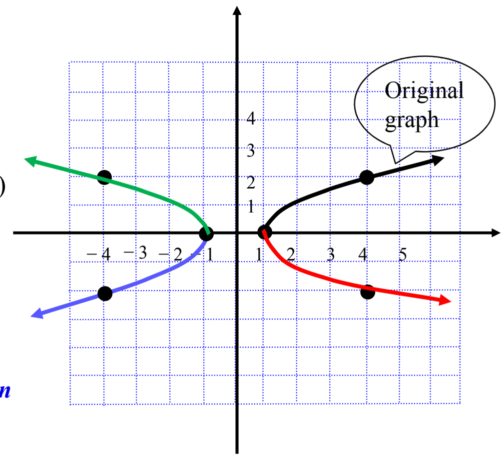
The image *of* $R(-2, 3)$ w.r.t. the *origin* is $(2, -3)$



Ex 3 Sketch a graph that is symmetric to the given graph with respect to the x-axis, y-axis, and the origin

Sol

Lets identify some points on the graph that can help us to achieve our goals, such as (1, 0), (4, 2)….



To obtain a graph symmetric w.r.t the ***x-axis***

(1, 0) will be imaged to (1, – 0) (itself)

(4, 2) will be imaged to (4, – 2)

To obtain a graph symmetric w.r.t the ***y-axis***

(1, 0) will be imaged to (–1, 0)

(4, 2)Will be imaged to (– 4, 2)

To obtain a graph symmetric w.r.t the ***origin***

(1, 0) will be imaged to (– 1, – 0)

(4, 2) Will be imaged to (– 4, – 2)

II) Even and Odd Functions

A function f is **even** if $f(-x) = f(x)$ for every x in the domain of f.

A function f is **odd** if $f(-x) = -f(x)$ for every x in the domain of f.

Ex 4 Determine whether each of the following functions is even, odd , or neither.

a) $f(x) = x^2$

Sol

$$f(-x) = (-x)^2 = x^2 \text{ which is equal to } f(x)$$

$$\Rightarrow f(x) = f(-x)$$

Thus $f(x)$ is an even function.

$b)\quad g(x) = x^3$

Sol

$g(-x) = (-x)^3 = -x^3$ which is equal to $-\ g(x)$

$\Rightarrow g(-x) = -g(x)$ Thus, $g(x)$ is an odd function.

$c)\quad h(x) = x^3 + x^2$

Sol

$h(-x) = (-x)^3 + (-x)^2 = -x^3 + x^2$

$\Rightarrow h(x) \neq \begin{cases} h(-x) \\ -h(x) \end{cases}$ Thus $h(x)$ is neither even nor odd

Practice:

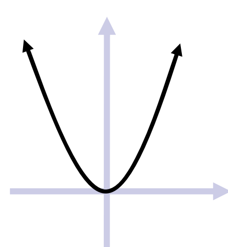
$d)\quad k(x) = 5$ — even

$e)\quad f(x) = \dfrac{x^3}{x^2+1}$ — odd

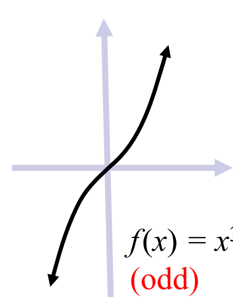
$f)\quad f(x) = \dfrac{x^4}{\sqrt[5]{x^3 - x}}$ — odd

Notes:

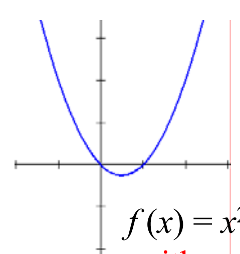
- The graph of an **even** function is **symmetric** with respect to with respect to the y-axis.
- The graph of an **odd** function is **symmetric** with respect to with respect to the **origin**.
- If the graph is **not symmetric** with respect to the **y-axis** or the **origin**, then function is **neither even nor odd**



$f(x) = x^2$ (even)



$f(x) = x^3$ (odd)



$f(x) = x^2 - x$ neither

III) Translation (الإنسحاب) of Graphs:

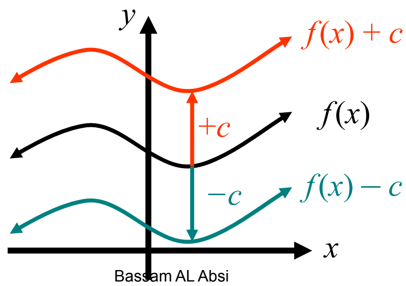
(A) Vertical Translations:

If c is a ***positive*** real number, the graph of $y - c = f(x)$ is the graph of $y = f(x)$ **shifted upward** c units. (*y is replaced by* $\boldsymbol{y - c}$)

If c is a ***positive*** real number, the graph of $y + c = f(x)$ is the graph of $y = f(x)$ **shifted downward** c units. (*y is replaced by* $\boldsymbol{y + c}$)

Notice that $y - c = f(x)$ can be written as $y = f(x) + c$

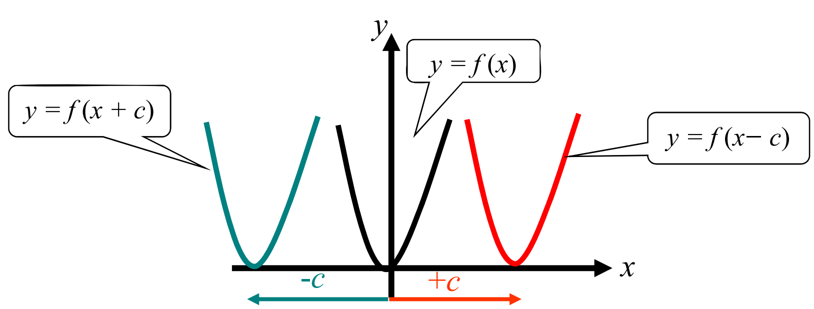
Notice that $y + c = f(x)$ can be written as $y = f(x) - c$



(B) Horizontal Translations :

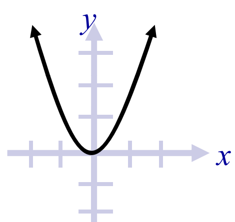
If c is a **positive** real number, then the graph of $f(x - c)$ is the graph of $y = f(x)$ **shifted to the right** c units. (*x is replaced by* $\boldsymbol{x - c}$)

If c is a **positive** real number, then the graph of $f(x + c)$ is the graph of $y = f(x)$ **shifted to the left** c units. (*x is replaced by* $\boldsymbol{x + c}$)

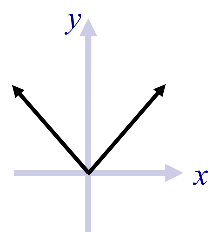


Library of Mother Graphs:

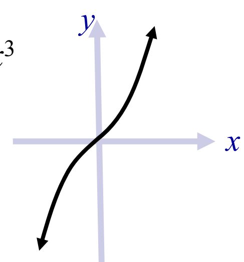
1) $y = x^2$



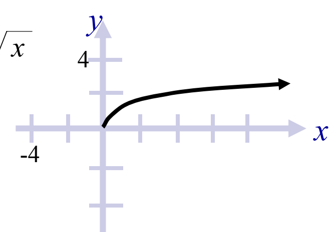
2) $y = |x|$



3) $y = x^3$

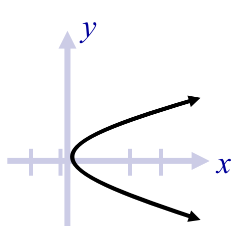


4) $y = \sqrt{x}$

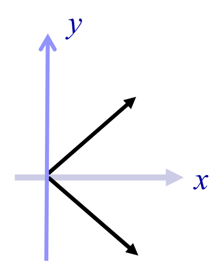


Library of Mother Graphs:

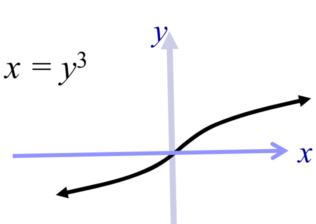
1) $x = y^2$



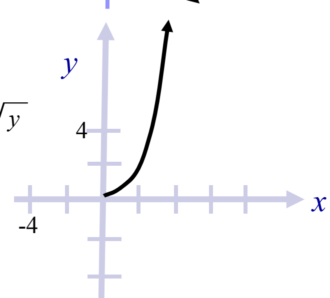
2) $x = |y|$



3) $x = y^3$



4) $x = \sqrt{y}$



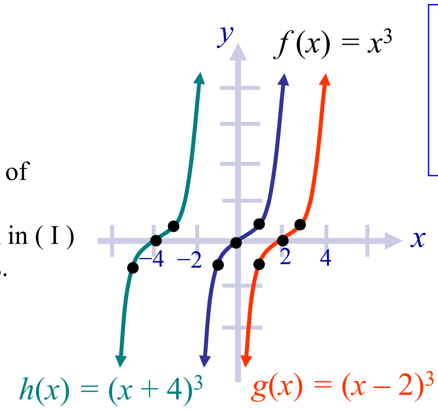
Ex 5 Use the graph of $f(x) = x^3$ to graph $g(x) = (x - 2)^3$ and

$h(x) = (x + 4)^3$.

Sol

I) Sketch the graph of $f(x) = x^3$

II) Sketch the graph of $g(x) = (x - 2)^3$ by shifting the graph in (I) to the right two units.



III) Sketch the graph of $h(x) = (x + 4)^3$ by shifting the graph (I) to the left four units.

Ex 6 Use the graph of $f(x) = |x|$ to graph the functions

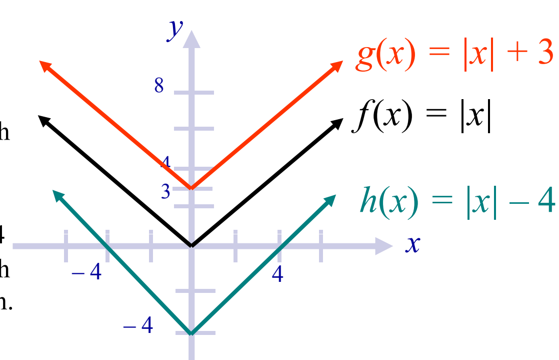
$g(x) = |x| + 3$ and $h(x) = |x| - 4$.

Sol

I) graph $f(x) = |x|$

II) graph $g(x) = |x| + 3$ by shifting the graph in (I) 3 units up..

III) graph $h(x) = |x| - 4$ by shifting the graph in (I) 4 units down.



Ex 7 Find the range of the function $f(x) = |x+1| - 3$

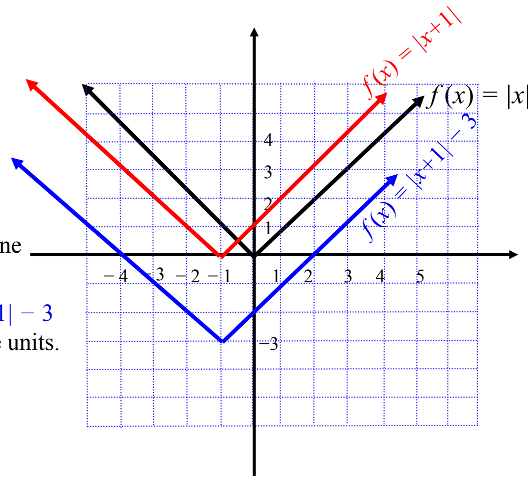
Sol

Lets do this problem using the graphical technique as follows:

I) Sketch the graph of $f(x) = |x|$

II) Sketch the graph of $f(x) = |x+1|$ by shifting the graph (I) to the left one unit.

III) Sketch the graph of $f(x) = |x+1| - 3$ by shifting the graph (II) down three units.



Thus, the range is $[-3, \infty)$

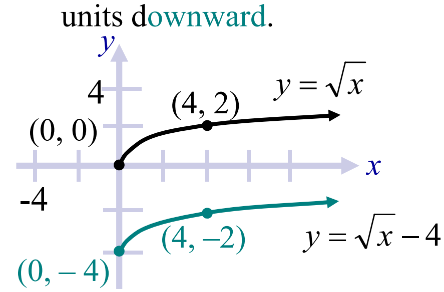
Ex 8 Graph the function $y = \sqrt{x+5} - 4$ using the graph of $y = \sqrt{x}$

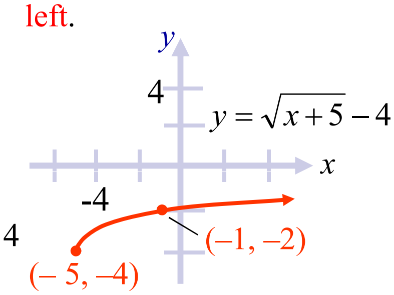
Sol

(I) graph $y = \sqrt{x}$

(II) graph $y = \sqrt{x} - 4$ by shifting the graph in (I) 4 units downward.

(III) Graph $y = \sqrt{x+5} - 4$ by shifting the graph in (II) 5 units left.





Ex 9 If the graph of the function $y = \frac{x-1}{x+3}$ is shifted horizontally two units to the left and vertically two units up, then find the equation of the new graph.

Sol Since the graph of $y = \frac{x-1}{x+3}$ is:

Shifted horizontally two units to the left: **replace x by $x+2$**

Shifted vertically two units to up: **replace y by $y-2$**

The equation of the resulting graph is

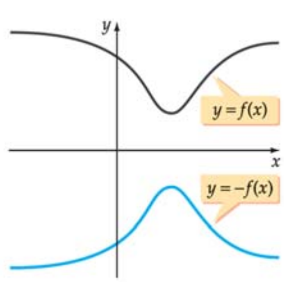
$$y-2 = \frac{(x+2)-1}{(x+2)+3} \qquad y-2 = \frac{x+1}{x+5}$$

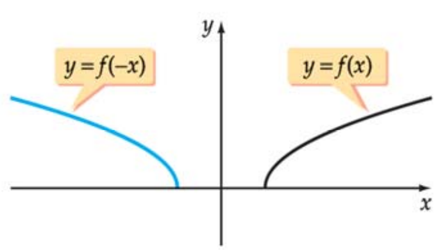
The equation of the resulting graph is $y = \frac{3x+11}{x+5}$

IV) Reflections (الإنعاس)of Graphs

•The graph of the function $y = -f(x)$ is the graph of $y = f(x)$ **reflected across the *x*-axis**.

• The graph of the function $y = f(-x)$ is the graph of $y=f(x)$ **reflected across the *y*-axis**.





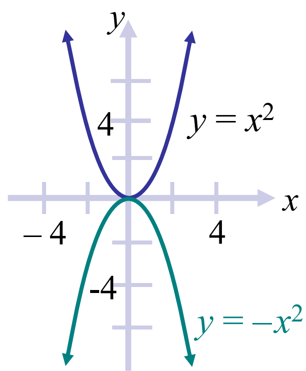
Ex 10 Graph $y = -(x+3)^2$ using the graph of $y = x^2$.

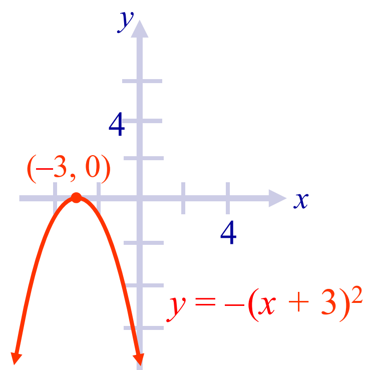
Sol

(I) graph $y = x^2$

(II) *reflect* the graph (I) in the x-axis.

(III) *shift* the graph (II) three units to the left.





V) Stretching (التمدد)and Compressing (التقلص)of Graphs

I) Vertical Stretching and Compressing

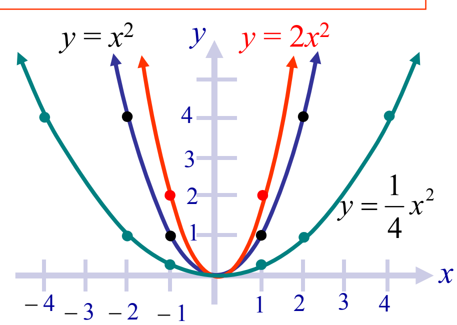
If $c > 1$ then the graph of $y = c\,f(x)$ is the graph of $y = f(x)$ **stretched vertically by *c*.**

If $0 < c < 1$ then the graph of $y = c\,f(x)$ is the graph of $y = f(x)$ **shrunk vertically by *c*.**

Ex 11

$y = 2x^2$ is the graph of $y = x^2$ *stretched vertically* by 2.

$y = \frac{1}{4}x^2$ is the graph of $y = x^2$ *compressed vertically* by $\frac{1}{4}$.



II) Horizontal Stretching and Shrinking

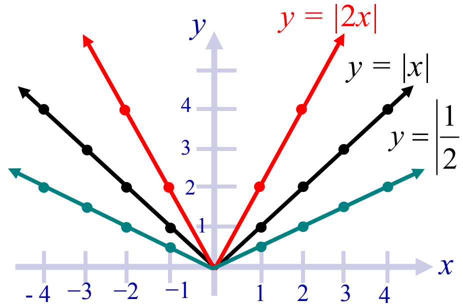
If $c > 1$, the graph of $y = f(c\,x)$ is the graph of $y = f(x)$ **shrunk horizontally by 1 / c**.

If $0 < c < 1$, the graph of $y = f(c\,x)$ is the graph of $y = f(x)$ **stretched horizontally by 1 / c**.

Ex 12

$y = |2x|$ is the graph of $y = |x|$ *shrunk horizontally* by 1/2.

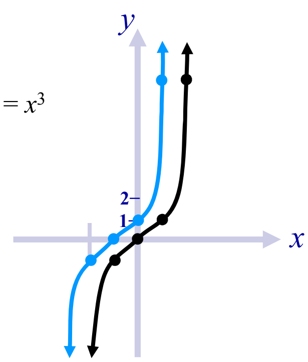
$y = \left|\frac{1}{2}x\right|$ is the graph of $y = |x|$ *stretched horizontally* by 2 .



Ex 13 Graph $y = \frac{1}{2}(x+1)^3 + 3$ given the graph $y = x^3$.

Sol

(I) graph $y = x^3$

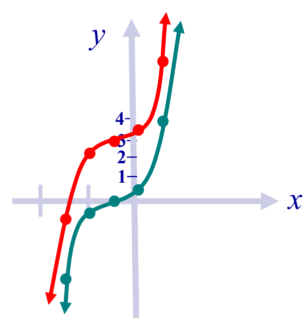


(II) graph $y = (x + 1)^3$

Shift (I) 1 unit to the left

(III) graph $y = \frac{1}{2}(x+1)^3$

shrink (II) ½ unit



(IV) graph $y = \frac{1}{2}(x+1)^3 + 3$

shift (III) 3 units up

Ex 14 Graph $y = -3|x+1| - 2$ using the graph $y = |x|$.

Sol

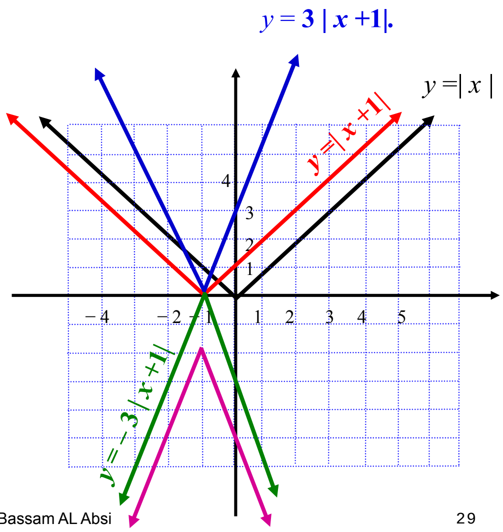
(I) graph $y = |x|$.

(II) graph $y = |x+1|$.
shift the graph in (I) 1 unit to the left

(III) graph $y = 3|x+1|$.
shrink the graph in (II) by multiplying every y value by 3

(IV) graph $y = -3|x+1|$.
Reflect the graph in (III) about the x-axis

(V) graph $y = -3|x+1| - 2$.
shift the graph in (IV) down 1 unit



Ex 15 If the point $(-1, 2)$ lies on the graph of $y = f(x)$, then find the image of this point on the graph of the following functions:

Sol

1) $y = 2f(x)$ the emage is $(-1, 2\cdot 2) = (-1, 4)$

2) $y = f(2x)$ the emage is $(\frac{1}{2}\cdot(-1), 2) = \left(-\frac{1}{2}, 2\right)$

3) $y = 2f(\frac{1}{3}x)$ the emage is $(3\cdot(-1), 2\cdot 2) = (-3, 4)$

4) $y = 3f(x) + 1$ the emage is $(-1, 3\cdot 2 + 1) = (-1, 7)$

5) $y = -3f(2x) + 2$ the emage is $(\frac{1}{2}\cdot -1, -3\cdot 2 + 2) = \left(-\frac{1}{2}, -4\right)$