2.4 QUADRATIC FUNCTIONS

Objectives:

In this section, we learn about:

- Writing the equation of a parabola in standard from
- Sketching the graph of a parabola, knowing its:
  - domain.
  - Vertex.
  - Axis of symmetry.
  - x and y intercepts.
  - Range.
  - Increasingness and decreasingness intervals.
- Applications.

Definition:

- Let a, b, and c be real numbers $a \neq 0$. The function

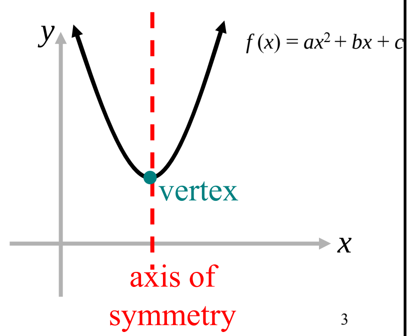
$$f(x) = ax^2 + bx + c$$

is called a **quadratic function**.

•The graph of a quadratic function is a ***vertica*l parabola** (قطع مكافي) .

•Every parabola is symmetric about a line called the **axis of symmetry** (محور التناظر).

• The intersection point of the parabola and the axis of symmetry is called the **vertex** (رأس) of the parabola.



Graph of a Quadratic Function:

The quadratic function $f(x) = ax^2 + bx + c$ can be written (*using completing the square*) in the **standard form**

$$\boldsymbol{f(x) = a(x - h)^2 + k} \quad (a \neq 0)$$

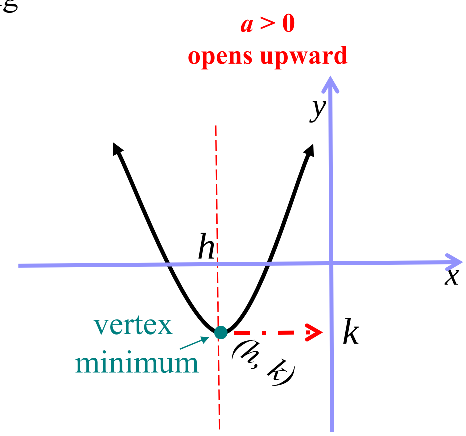
where

$$\boldsymbol{h = -b/2a \quad and \quad k = f(h) = f(-b/2a)}$$

The domain of f *is all real numbers, unless otherwise stated.*

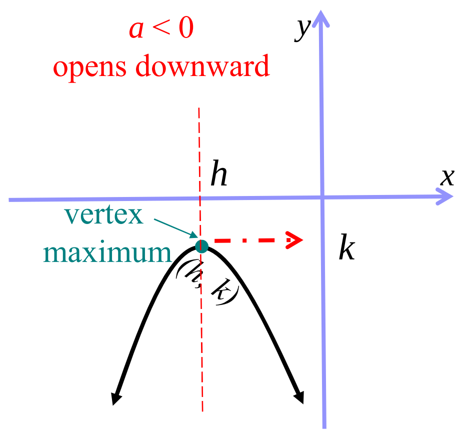
The graph of f has the following characteristics:

Case 1:



- **Vertex** (h, k).
- axis of symmetry $x = h$
- Minimum value of y is k
- Range: $[k, \infty)$
- Increasing: (h, ∞)
- Decreasing: $(-\infty, h)$
- The **y-intercept** is $f(0) = c$.
- The x-intercepts are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, *if* $b^2 - 4ac \geq 0$

Case2: a < 0



- **Vertex** (h, k).
- axis of symmetry $x = h$
- Maximum value of y is k
- Range: $(-\infty, k]$
- Increasing: $(-\infty, h)$
- Decreasing: (h, ∞)
- The **y-intercept** is $f(0) = c$.
- The x-intercepts are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, *if* $b^2 - 4ac \geq 0$

Ex1 Consider the function $f(x) = 2x^2 + 4x + 3$

Write the function in the standard form.

1. Find the vertex.
2. Find the axis of symmetry.
3. Find , if any, the maximum value of the function.
4. Find , if any, the minimum value of the function.
5. Find the range of the function.
6. Find the interval(s) of increasing and decreasing.
7. Sketch the graph of the function and show on the graph the intercept(s), the vertex, and the axis of symmetry.

a) $f(x) = 2x^2 + 4x + 3$

Sol

1. $$h = -\frac{b}{2a} = \frac{-4}{2(2)} = -1$$

$$k = f(h) = f(-1) = 1$$

The standard form is:

$$f(x) = a(x-h)^2 + k$$
$$= 2(x+1)^2 + 1$$

So the equation is

$$\boxed{f(x) = 2(x-(-1))^2 + 1}$$

h k

2. The vertex is $(h, k) = (-1, 1)$

3. The axis of symmetry: $x = h$

$$x = -1$$

4. $f(x)$ has no maximum value(since it opens upwards).

5. The minimum value of is $k = 1$.

6. The range= $[k, \infty) = [1, \infty)$

7. $f(x)$ is increasing on $(h, \infty) = (-1, \infty)$

$f(x)$ is decreasing on $(-\infty, \boldsymbol{h}) = (-\infty, -1)$

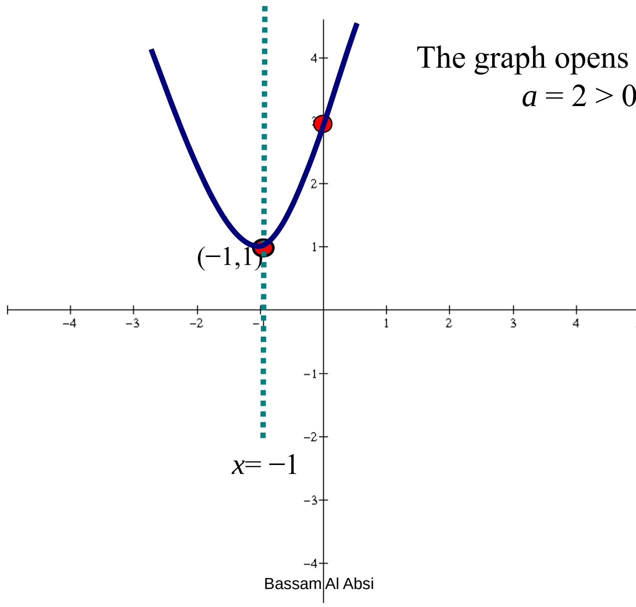
8. <u>x-intercept(s):</u>

$$\boldsymbol{y} = 0 \Rightarrow 2\boldsymbol{x}^2 + 4\boldsymbol{x} + 3 = 0 \Rightarrow \text{no } \boldsymbol{real} \text{ solutions}$$
$$\Rightarrow \text{no } \boldsymbol{x} - \text{intercept}$$

<u>y-intercept:</u>

$x = 0 \Rightarrow y = 3 \Rightarrow$ the y-intercept is (0, 3)

Hence ,the graph of $f(x)$ is as shown below:



The graph opens up since $a = 2 > 0$

Ex 2 Use completing the square to graph, the following function, find the vertex and y and x-intercepts of $f(x) = -x^2 + 6x + 7$.

$f(x) = -x^2 + 6x + 7$ original equation

$f(x) = -(x^2 - 6x) + 7$ factor out –1

$f(x) = -(x^2 - 6x + 9) + 7 + 9$ complete the square

$f(x) = -(x - 3)^2 + 16$ standard form

$a < 0 \rightarrow$ parabola opens downward.

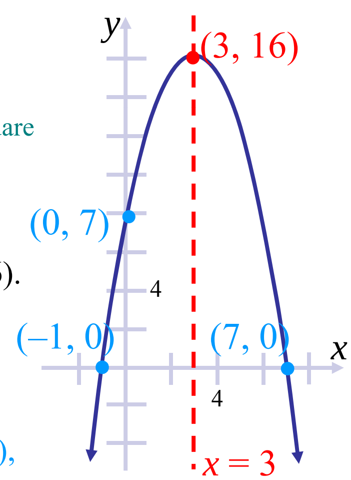
$h = 3, k = 16 \rightarrow$ axis $x = 3$, vertex $(3, 16)$.

Find the x-intercepts by solving $-x^2 + 6x + 7 = 0$.

$(-x + 7)(x + 1) = 0$ factor

$x = 7, x = -1$ y-intercepts (0,7),

x-intercepts (7, 0), (–1, 0)



$f(x) = -x^2 + 6x + 7$

Ex 3 Find the quadratic function in x whose graph has ***(attains)*** its minimum value at (2, 1) and passes through the point (0, 4)

Sol The equation, ***in standard form***, is $f(x) = a(x-h)^2 + k$

Since the quadratic function graph attains it **minimum at the vertex**

then , $(h, k) = (2, 1)$ $\Rightarrow f(x) = a(x-2)^2 + 1$

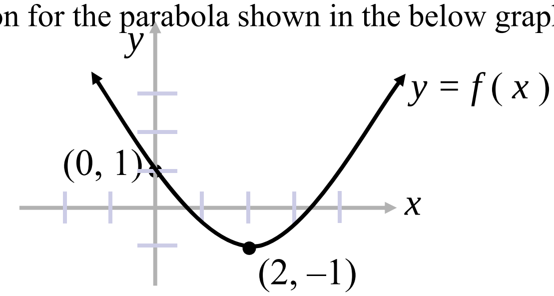
the graph passes through the point (0, 4)

Substitute $x = 0$ and $y = 4$ $\Rightarrow 4 = a(0-2)^2 + 1$

$\Rightarrow a = \frac{3}{4}$ Thus, the quadratic function is

$$\Rightarrow f(x) = \frac{3}{4}(x-2)^2 + 1$$

Ex 4 Find an equation for the parabola shown in the below graph.



Sol

$f(x) = a(x-h)^2 + k$, (**standard form)**

$f(x) = a(x-2)^2 + (-1)$, **vertex** $(2, -1) = (h, k)$

Since (0, 1) is a point on the parabola: $f(0) = a(0-2)^2 - 1$

$1 = 4a - 1$ and $a = \frac{1}{2}$

$$f(x) = \frac{1}{2}(x-2)^2 + (-1) \Rightarrow \quad f(x) = \frac{1}{2}x^2 - 2x + 1$$

Ex 5 Find two numbers whose sum is 8 and whose product is a maximum.

Sol Let the numbers be x and y

Their sum is 8: $x + y = 8 \quad (\mathbf{1})$

Their product: $p = x\,y \quad (2)$

Now, we need p to be maximum ?

To achieve this goal, write p as a function in ONE variable ?

Notice that from (**1**) , $\Rightarrow \boldsymbol{x} = 8 - \boldsymbol{y}$

$$\Rightarrow \boldsymbol{p} = \boldsymbol{y}(8 - \boldsymbol{y}) = -\boldsymbol{y}^2 + 8\boldsymbol{y}$$

$$\Rightarrow \boldsymbol{y} = \boldsymbol{h} = -\frac{8}{2(-1)} = 4$$

$$\boldsymbol{and} \;\; \boldsymbol{x} = 8 - 4 = 4$$

The graph of p is a parabola that opens ***downward***. It attains its ***max.*** area at the ***vertex*** of its graph.

Ex 6 If x is a real number, then find the maximum area of a rectangle of length $3+2x$ and width $1-2x$.

Sol

$$\text{Area} = (3 + 2x)(1 - 2x)$$

$$\boldsymbol{A}(\boldsymbol{x}) = -4\boldsymbol{x}^2 - 4\boldsymbol{x} + 3$$

The area is max. if $\boldsymbol{x} = \boldsymbol{h} = -\dfrac{-4}{2(-4)} = -\dfrac{1}{2}$

Thus, the max. area is

$$\boldsymbol{k} = \boldsymbol{A}\left(\frac{-1}{2}\right) = -4\left(\frac{-1}{2}\right)^2 - 4\left(\frac{-1}{2}\right) + 3$$

$$= 4 \;\boldsymbol{units\ square}$$

The graph of $A(x)$ is a parabola that opens ***downward.*** It attains its ***max.*** area at the ***vertex*** of its graph.

Ex 7 If $x = 3$ is the axis of symmetry of the parabola

$y = -2x^2 + cx + 2$, then find c .

Sol The axis of symmetry of the parabola is

$$x = h = \frac{-b}{2a}$$

$$\Rightarrow \ 3 = \frac{-c}{2(-2)}$$

$$\Rightarrow \ c = 12$$

Ex 8 A basketball is thrown from the free throw line from a height of six feet. What is the maximum height of the ball if the path of the ball is given by: $-3x^2 + 4x + 2$

Sol $$Let \ \ H(x) = -3x^2 + 4x + 2$$

The graph of $h(x)$ is a parabola that opens downward. It attains its max. height at the vertex of its graph.

$$x = h = \frac{-b}{2a} = \frac{-4}{2(-3)} = \frac{2}{3}$$

Thus, the max. height is,

$$k = H\left(\frac{2}{3}\right) = -3\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right) + 2 = \frac{10}{3} \ \ ft$$

Ex 9 Use completing the square technique to graph the parabola $g(x) = -3x^2 - 2x + 1$

Sol Factor out -3 from the **x terms**

$$f(x) = -3\left(x^2 + \frac{2}{3}x \qquad \right) + 1$$

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right) + 1$$

$$f(x) = -3\left(x^2 + \frac{2}{3}x + \frac{1}{9}\right) + 1 + \frac{1}{3}$$

$$f(x) = -3\left(x + \frac{1}{3}\right)^2 + \frac{4}{3}$$

thus the equation becomes $f(x) = -3\left(x - \left(-\frac{1}{3}\right)\right)^2 + \frac{4}{3}$

The axis of symmetry is the vertical line $x + \frac{1}{3} = 0$

Vertex $\left(-\frac{1}{3}, \frac{4}{3}\right)$

y – intercept is $y = -3(0)^2 - 2(0) + 1 = 1$

x – intercept is

$-3x^2 - 2x + 1 = 0$

$3x^2 + 2x - 1 = 0$

$x = -1, 1/3$