2.3 LINEAR FUNCTIONS

Objectives:

In this section, we will learn about:

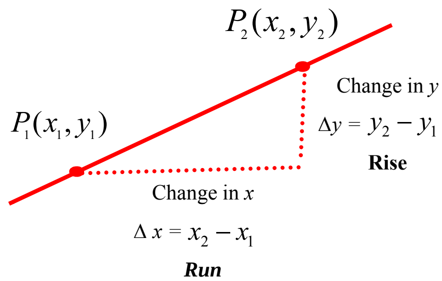
- Slope of Lines.
- Slope-Intercept Form.
- Finding the Equation of a Line.
- Parallel and Perpendicular Lines.

Def:

A function of the form $f(x) = mx + b,\ m \neq 0$ where m and b are real numbers is called a **linear function**

- **Slopes of Lines (الميل)**

The slope is a numerical measure of the **steepness** of a line.

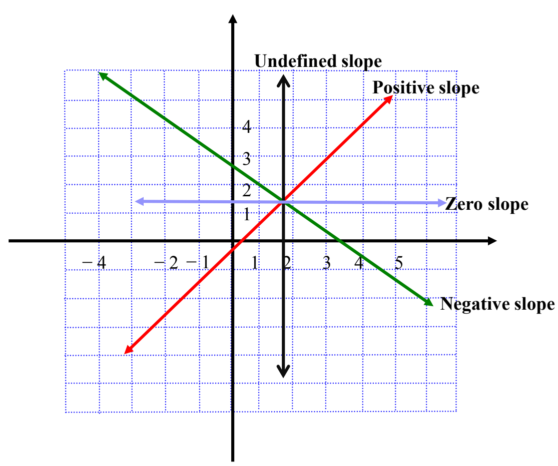


The slope = $\dfrac{rise}{run}$

$$\text{The slope} = m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1},\ (x_1 \neq x_2).$$

- The slope of a line can be positive, negative, undefined, or zero, as shown in the adjacent figure.

- The slope is **independent** of the choice of points on the line.



Ex 1 Find the slope of the line that passes through the following points $(-3, 4)$ and $(1, -2)$

Sol Let $(x_1, y_1) = (-3, 4)$ and $(x_2, y_2) = (1, -2)$

$$\text{The slope} = m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2-4}{1-(-3)}$$

$$= \frac{-6}{4} = \frac{-3}{2}$$

Ex 2 Sketch the graph of the line that has slope $\frac{-2}{3}$ and passes through the point $(1, -2)$

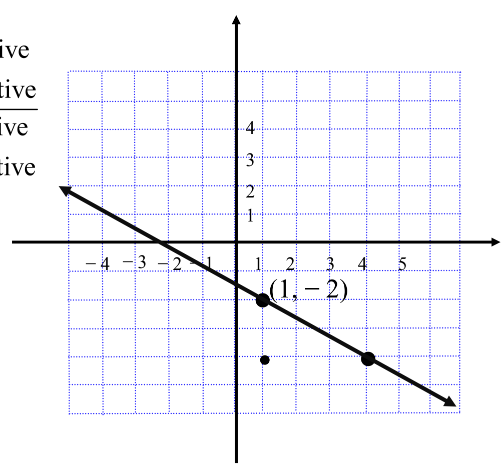
Sol

$$\text{Slope} = \frac{\text{rise}\begin{cases}\text{move up} & \text{if positive}\\ \text{move down} & \text{if negative}\end{cases}}{\text{run}\begin{cases}\text{move right} & \text{if positive}\\ \text{move left} & \text{if negative}\end{cases}}$$

Locate the point $(1, -2)$

$$\text{Slope} = \frac{-2}{3}$$

$$= \frac{\text{move down 2 units}}{\text{move to the right 3 units}}$$



The Equation of a Line :

Case1: Point - Slope Form :

The equation of a line with **slope *m*** and passing **through** the point (x_1, y_1) is given by

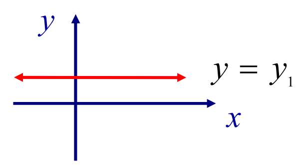
$$y - y_1 = m(x - x_1)$$

Case2: Horizontal Line:

The equation of **horizontal line** passing **through** the point

(x_1, y_1) is give by

$y = y_1$ (slope =0)

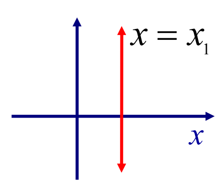


Case3: Vertical Line:

The equation of **vertical line** passing **through** the point

(x_1, y_1) is give by is given by

$x = x_1$ (slope is undefined)



Case4: Slope-Intercept Form:

The equation of a line with **slope *m*** and **y-intercept** $(0, b)$ is given by $y = m\,x + b$.

Important Note:

If $y = m\,x + b$, then the **slope** is the **coefficient of *x* .**

Ex 3 Find the equation of the straight lines that have the following properties. Write the answer in the form $y = mx + b$

a) has slope 2 / 3 and passes through the point (− 3, 4)

Sol

Let $m = 2/3$ and $(x_1, y_1) = (-3, 4)$

The equation is $y - y_1 = m(x - x_1)$

Upon substitution, we get $y - 4 = \frac{2}{3}(x - (-3))$

$\Rightarrow y - 4 = \frac{2}{3}(x + 3)$ $\qquad \Rightarrow 3y - 12 = 2x + 6$

$\Rightarrow 3y = 2x + 18$ $\qquad \Rightarrow$ the equation is $y = \frac{2}{3}x + 6$

b) passes through the points (5, - 6) and (2, -8)

Sol Let $(x_1, y_1) = (5, -6)$ and $(x_2, y_2) = (2, -8)$

again, the equation is $y - y_1 = m(x - x_1)$

$$m = \frac{-8 - (-6)}{2 - 5} = \frac{2}{3}$$

Upon substitution, we get $y - (-6) = \frac{2}{3}(x - 5)$

$\Rightarrow 3y + 18 = 2x - 10$

$\Rightarrow$ the equation is $y = \frac{2}{3}x - \frac{28}{3}$

c) has undefined slope and passes through the point $(-2, 3)$

Sol

This described line is **vertical** and its equation is $x = -2$

d) horizontal and passes through the point $(1/2, 4/7)$

Sol

This line has zero slope and its equation is $y = 4/7$

Ex 3 If $f(x)$ is a linear function with $f(5) = -6$ and $f(2) = -8$, find $f(-2)$.

Sol

Remember: $y = f(x)$

Give $f(5) = -6$ and $f(2) = -8$

Let $(x_1, y_1) = (5, -6)$ and $(x_2, y_2) = (2, -8)$

Thus, the equation is $y - y_1 = m(x - x_1)$

$$m = \frac{-8-(-6)}{2-5} = \frac{2}{3}$$

Upon substitution, we get $y-(-6) = \frac{2}{3}(x-5)$

$\Rightarrow 3y + 18 = 2x - 10 \quad \Rightarrow$ the equation is $y = \frac{2}{3}x - \frac{28}{3}$

or $f(x) = \frac{2}{3}x - \frac{28}{3} \quad \Rightarrow \quad f(-2) = \frac{2}{3}(-2) - \frac{28}{3} = \frac{-32}{3}$

Parallel (متوازي) and Perpendicular (متعامد) Lines

Consider the two lines

$$l_1 : m_1 x + b_1 \quad \text{and} \quad l_2 : \ m_2 x + b_2$$

1) l_1 and l_2 are parallel if and only if $m_1 = m_2$

2) l_1 and l_2 are perpendicular if and only if $m_1 = -\dfrac{1}{m_2}$

$[\text{OR} \quad m_1 \cdot m_2 = -1]$

Ex 4 Determine whether the two lines whose equations are

$$2x + 3y = 1 \quad \text{and} \quad 3x - 2y = 4$$

are parallel or perpendicular.

Sol

Let $l_1 : 2x + 3y = 1$ and $l_2 : \ 3x - 2y = 4$

Now find the slope of each line as described in the adjacent rectangle:

$$m_1 = \frac{-2}{3} \quad \text{and} \quad m_2 = \frac{-3}{-2} = \frac{3}{2}$$

notice that $m_1 \cdot m_2 = -1$

Thus, the two lines are perpendicular.

If a line has equation $\boldsymbol{A\,x + By = C}$, then its **slope** is equal to $\dfrac{-A}{B}$

Ex 5 Find the equation of a line that passes through the point (1, 3) and parallel to the line $3x + 4y = 4$.

Sol

Let l_1 : *be* the line passing thru $(x_1, y_1) = (1,3)$

and l_2 : $3x + 4y = 4$

The equation of l_1 that we need to find is $(y - y_1) = m_1(x - x_1)$

The slope of l_1 is m_1 which is equal to m_2 ?

$$m_2 = \frac{-3}{4} \underset{\text{are parallel}}{\overset{\text{since } l_1 \text{ and } l_2}{=}} m_1$$

Thus the equation is $(y-3) = \frac{-3}{4}(x-1) \Rightarrow y = \frac{-3}{4}x + \frac{15}{4}$

Ex 6 Find the equation of a line that passes through the point (−3, 4) and perpendicular to the line $2x - y = 7$.

Sol

Let l_1 : *be* the line passing thru $(x_1, y_1) = (-3,4)$

and l_2 : $2x - y = 7$

The equation of l_1 that we need to find is $(y - y_1) = m_1(x - x_1)$

The slope of l_1 is m_1 which is equal to $\frac{-1}{m_2}$?

$$m_2 = 2 \underset{\text{are perpendicular}}{\overset{\text{since } l_1 \text{ and } l_2}{=}} \frac{-1}{m_1} \Rightarrow m_1 = -\frac{1}{2}$$

Thus the equation is $(y-4) = \frac{-1}{2}(x+3)$

Ex 7 If the two lines $3x - 5y + 1 = 0$ and $k\,y - 2x + 4 = 0$ are perpendicular, then find k.

Sol

Let $l_1 :\ 3x - 5y + 1 = 0$ and $l_2 :\ k\,y - 2x + 4 = 0$

Since the two lines are perpendicular, then:

$$m_1 = \frac{-1}{m_2} \Rightarrow \frac{-3}{-5} = \frac{-1}{\frac{-(-2)}{k}}$$

$$\Rightarrow \frac{3}{5} = \frac{-k}{2} \Rightarrow k = \frac{-6}{5}$$

Ex 8 If (2,3) , (4,2) and (k,0) are three points on the same straight line , then find k.

Sol

Since the three points are on the same line, the **slope** between **any two points is fixed.**

The slope using the points (2,3) , (4,2) is $\frac{2-3}{4-2} = \frac{-1}{2}$

and the slope using the points (4,2) and (k,0) is $\frac{0-2}{k-4} = \frac{-2}{k-4}$

and since the slope is fixed, then $\frac{-1}{2} = \frac{-2}{k-4}$

$$\Rightarrow k - 4 = 4 \Rightarrow k = 8$$

Ex 9 Find the value of x in the domain of $f(x) = 4x - 3$ for which $f(x) = -2$.

Sol since $f(x) = -2$, using substitution, we get:

$-2 = 4x - 3 \qquad x = 1/4$

Ex 10 Find a point $P(x, y)$ on the graph of the equation $y = x^2$ such that the slope of the line through the point (3, 9) and P is 15 / 2.

Sol Since the slope of the line through the point (3, 9) and $P(x, y)$ is 15/2, then:

$\frac{y-9}{x-3} = \frac{15}{2}$, and since $P(x, y)$ is on the graph of the equation $y = x^2$, **then:**

$$\frac{x^2-9}{x-3} = \frac{15}{2} \Rightarrow x+3 = \frac{15}{2} \Rightarrow x = \frac{9}{2}$$

Thus, the point is (9/ 2 , 81 / 4)

Ex 11 Find all points (x, y) satisfying $x + y = 0$, that are 6 units from the points $(-3, 4)$.

Sol The set of all points (x, y) that are 6 units from the points $(-3, 4)$ represents the **circle** that has center at $(-3, 4)$ and radius $r = 6$ and equation is $(x+3)^2 + (y-4)^2 = 36$

$$\Rightarrow x^2 + 6x + 9 + y^2 - 8y + 16 = 36 \ldots\ldots (*)$$

the point(s) (x, y) satisfying $x + y = 0$

this implies that $x = -y$, substitute in Eq. (*)

$$\Rightarrow (-y)^2 + 6(-y) + 9 + y^2 - 8y + 16 = 36$$

$$\Rightarrow 2y^2 - 14y - 11 = 0 \Rightarrow y = \frac{14 \pm 2\sqrt{71}}{4}$$

The points are $\left(\frac{-7-\sqrt{71}}{2}, \frac{7+\sqrt{71}}{2}\right)$ ***and*** $\left(\frac{-7+\sqrt{71}}{2}, \frac{7-\sqrt{71}}{2}\right)$

Point of Intersection (نقطة التقاطع)

To find the point of intersection between two lines solve the two equations simultaneously.

Ex 12 Find the point of intersection between to the two lines $2y - 3x = 12$ and $y + 2x + 1 = 0$.

Sol

$y + 2x + 1 = 0 \Rightarrow y = -2x - 1$ Substitute in the other equation

$$\Rightarrow 2(-2\boldsymbol{x} - 1) - 3\boldsymbol{x} = 12$$

$$-4\boldsymbol{x} - 2 - 3\boldsymbol{x} = 12$$

$$\Rightarrow \boldsymbol{x} = -2$$

$$\therefore y = -2(-2) + 1 = 3$$

Hence, the point of intersection is $(-2, 3)$

The previous question can be also solved graphically.