# 2.1 A TWO-DIMENSIONAL COORDINATE SYSTEM AND GRAPHS

## Objectives:

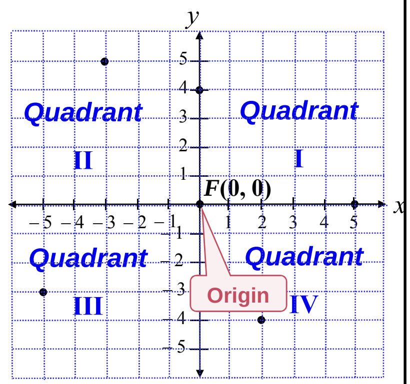
In this section, we will learn about:

- Cartesian Coordinate System.
- The Distance and Midpoint Formulas.
- Graph of an Equation.
- Intercepts.
- Circles:
  - Their Equations.
  - Their Graphs.

## Plotting Points in the Cartesian Coordinate System:

The Cartesian coordinate is a plane with the following features::

1) *Two axes:* ***x-axis (*** *horizontal real numbers) and* ***y-axis*** *( vertical real number).*
2) *The x-axis and y-axis intersect at the* ***Origin ( 0,0)****.*
3) *The plane is divided into four* ***quadrants****.*
4) *You can move through the plane by* ***points*** *of the form* ***( x, y).***
5) ***( x, y)*** *is called ordered pair.*
6) ***x*** *is called* ***the x -coordinate.***
7) ***y*** *is called* ***the y –coordinate.***






## Plotting Ordered Pairs:

*A*(– 3, 5): 3 units left, 5 units up

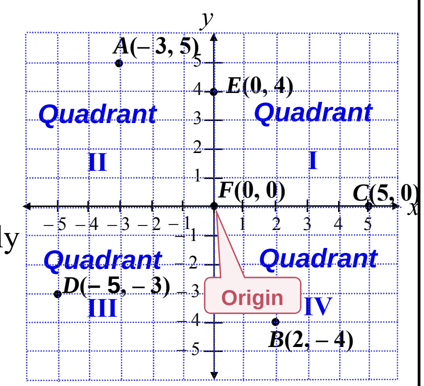
*B*(2, – 4): 2 units right, 4 units down

*C*(5, 0): 5 units right, 0 units vertically

*D*(– 5,– 3): 5 units left, 3 units down

*E*(0, 4): 0 units horizontally, 4 units up

*F*(0, 0): 0 units horizontally, 0 units vertically



The ordered Pairs $(a,b)$ and $(c,d)$ are equal

If and only if

$a=c$ and $b=d$

For instance, $if\ (x,4)=(-2,y),\ then\ x=-2\ and\ y=4$



## Distance Between Two Points:

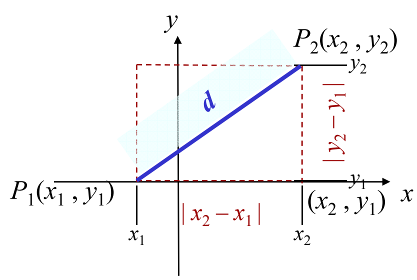
Consider the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the Cartesian coordinate system.

To find the distance $d$, use Pythagorean Theorem to get:

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$




Thus the distance between the two points $P_1$ and $P_2$ is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**Ex1** Find the distance between (– 1, – 3) and (2, 3).

**Sol:** Let $(x_1, y_1) = (-1, -3)$ and $(x_2, y_2) = (2, 3)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{[2-(-1)]^2 + [3-(-3)]^2}$$

$$d = \sqrt{(2+1)^2 + (3+3)^2}$$

$$d = \sqrt{3^2 + 6^2}$$

$$d = \sqrt{9+36}$$

$$d = \sqrt{45}$$

$$d = 3\sqrt{5} \approx 6.71$$



**Ex 2** Find the distance between the points $(x, 4x)$ and $(-2x, 3x)$, where $x < 0$.

**Sol:** Let $(x_1, y_1) = (x, 4x)$ and $(x_2, y_2) = (-2x, 3x)$,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{[-2x - x]^2 + [3x - 4x]^2}$$

$$d = \sqrt{(-3x)^2 + (-x)^2}$$

$$d = \sqrt{9x^2 + x^2}$$

$$d = \sqrt{10x^2}$$

$$d = \sqrt{10}\sqrt{x^2}$$

$$d = \sqrt{10}\,|x| \xrightarrow{\text{since } x<0} d = -\sqrt{10}\,x$$

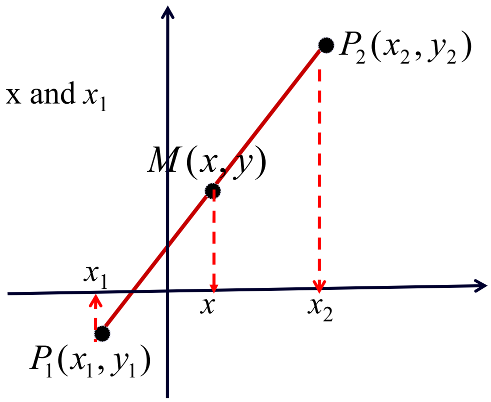
## The Midpoint Formula:

Consider the line segment whose endpoints are given by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as shown in the adjacent figure:

the distance between $x$ and $x_2$ equals to the distance between x and $x_1$ , that is $x_2 - x = x - x_1$

$$2\,x = x_2 + x_1$$

$$x = \frac{x_1 + x_2}{2}$$

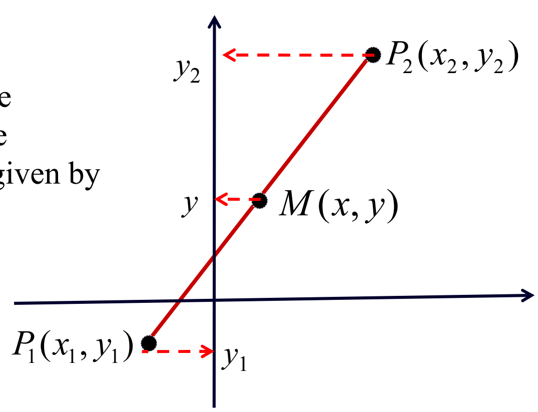





In a similar manner, the y-coordinate of the midpoint is $y = \frac{y_1 + y_2}{2}$

Thus, the midpoint of the line segment whose endpoints are $P_1(x_1 , y_1)$ and $P_2(x_2 , y_2)$ is given by

$$M = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$



**Ex 3** Find the midpoint of the line segment with endpoints (1, – 6) and (– 8, – 4).

**Sol:** Let $(x_1, y_1) = (1, -6)$ and $(x_2, y_2) = (-8, -4)$

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{1+(-8)}{2}, \frac{-6+(-4)}{2}\right)$$

$$= \left(\frac{-7}{2}, \frac{-10}{2}\right)$$

$$= \left(\frac{-7}{2}, -5\right)$$



**Ex 4** Find the other endpoint of the line segment that has endpoint (5,1) and midpoint (9,3).

**Sol:** Let one endpoint be $(x_1, y_1) = (5, 1)$, the midpoint be M = ( 9, 3 ), and the other endpoint be $(x_2, y_2) =$ ? , then:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$(9,3) = \left(\frac{5 + x_2}{2}, \frac{1 + y_2}{2}\right)$$

$$9 = \frac{5 + x_2}{2} \quad \text{and} \quad 3 = \frac{1 + y_2}{2}$$

$$\Rightarrow x_2 = 13 \quad \text{and} \quad y_2 = 5$$

the other endpoint is $= (13,5)$

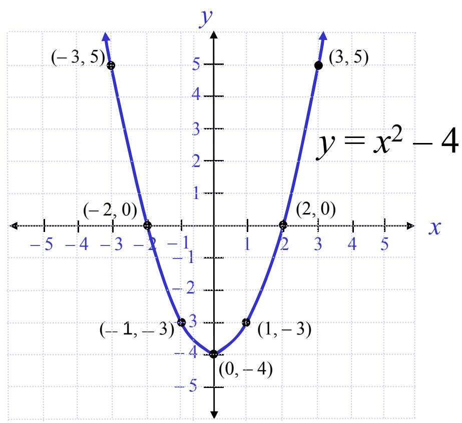
**Graphing an Equation :**

The graph of an equation in two variables $x$ and $y$ is the set of all points whose coordinates satisfy the equation.

**Ex 5** Graph $y = x^2 - 4$

**Sol:** Select integers for $x$:

| $x$ | $y$ | $= x^2 - 4$ |
|---|---|---|
| – 3 | 5 | $(-3)^2 - 4 = 5$ |
| – 2 | 0 | $(-2)^2 - 4 = 0$ |
| – 1 | –3 | $(-1)^2 - 4 = -3$ |
| 0 | –4 | $(0)^2 - 4 = -4$ |
| 1 | –3 | $(1)^2 - 4 = -3$ |
| 2 | 0 | $(2)^2 - 4 = 0$ |
| 3 | 5 | $(3)^2 - 4 = 5$ |



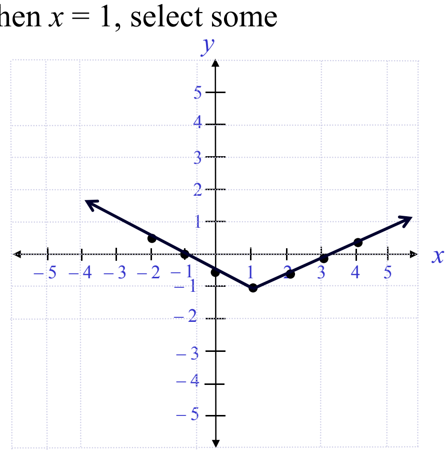



**Ex 6** Graph $2y + 2 = |x - 1|$

**Sol:** Solve for $y$ ( *if possible*): $y = \frac{1}{2}|x-1| - 1$

Remember that the absolute value function changes value around the point $x - 1 = 0$ or when $x = 1$, select some integers greater / less than 1

| $x$ | $y$ | $= ½ \|x-1\| - 1$ |
|---|---|---|
| – 2 | 1/2 | ½ \| – 3\| – 1 = 1/2 |
| – 1 | 0 | ½ \| – 2\| – 1 = 0 |
| 0 | –1/2 | ½ \| – 1\| – 1 = – 1/2 |
| 1 | – 1 | ½ \| 0\| – 1 = – 1 |
| 2 | – 1/2 | ½ \| 1\| – 1 = – 1/2 |
| 3 | 0 | ½ \| 2\| – 1 = 0 |
| 4 | ½ | ½ \| 3\| – 1 = 1/2 |



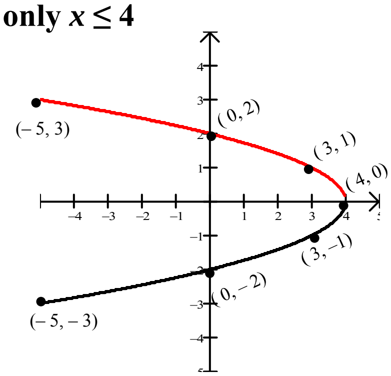
**Ex 7** Graph $y^2 + x = 4$

**Sol:** Solve for *y ( if possible)*: $y^2 = 4 - x$

$$y = \pm\sqrt{4-x}$$

For y to real valued, $4 - x$ must be nonnegative, i.e. $4 - x \geq 0$, this means that you can take **only $x \leq 4$**

| $x$ | $y = \pm\sqrt{4-x}$ | |
|---|---|---|
| 4 | 0 | $\pm\sqrt{4-4} = 0$ |
| 3 | $\pm 1$ | $\pm\sqrt{4-3} = \pm 1$ |
| 0 | $\pm 2$ | $\pm\sqrt{4-0} = \pm 2$ |
| $-1$ | $\pm\sqrt{5}$ | $\pm\sqrt{4-(-1)} = \pm\sqrt{5}$ |
| $-5$ | $\pm 3$ | $\pm\sqrt{4-(-5)} = \pm 3$ |






**Intercepts :**

An ***x-intercept*** of a graph is an **x-coordinate** of the point where the graph intersects the *x*-axis.

- If $(x_1, 0)$ satisfies an equation, then the point $(x_1, 0)$ is called an ***x-intercept*** of the equation.

The ***y-intercept*** of a graph is a **y-coordinate** of the point where the graph intersects the y-axis.

- If $(0, y_1)$ satisfies an equation, then the point $(0, y_1)$ is called an ***y-intercept*** of the equation.

**Ex 8** Find the $x$- and $y$-intercepts of the graph of $y = x^2 + 4x - 5$.

**Sol:**

- To find the $x$-intercepts, set $y = 0$

$0 = x^2 + 4x - 5$

$0 = (x - 1)(x + 5)$ Factor.

$x - 1 = 0 \qquad x + 5 = 0$

$x = 1 \qquad x = -5$

So, the $x$-intercepts are (1, 0) and (–5, 0).

- To find the $y$-intercept, set $x = 0$

$y = 0^2 + 4(0) - 5 = -5$

So, the $y$-intercept is (0, –5).

**Procedure for finding the $x$- and $y$- intercepts of the graph of an equation *algebraically*:**

- To find the $x$-intercepts of the graph of an equation, substitute 0 for $y$ in the equation and solve for $x$.
- To find the y-intercepts of the graph of an equation, substitute 0 for $x$ in the equation and solve for y.



**Ex 9** Find the $x$- and $y$-intercepts of the graph of $x = |y| - 2$

**Sol:**

- To find the $x$-intercepts, set $y = 0$

$$x = 0 - 2 = -2$$

So, the $x$-intercept is (−2, 0)

- To find the $y$-intercept, set $x = 0$

$$0 = |y| - 2$$

$$|y| = 2$$

$$y = \pm 2$$

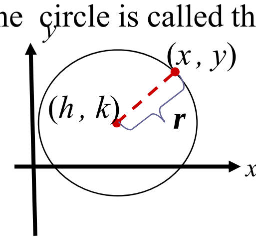
So, the $y$-intercepts are (0, 2) and (0, −2).

**Procedure for finding the $x$- and $y$- intercepts of the graph of an equation *algebraically*:**

- To find the $x$-intercepts of the graph of an equation, substitute 0 for $y$ in the equation and solve for $x$.
- To find the y-intercepts of the graph of an equation, substitute 0 for $x$ in the equation and solve for y.

## Definition of a Circle :

A ***circle*** is a set of all points in a plane that are at a fixed distance from a fixed point called the ***center*** $(h, k)$. The fixed distance from the circle's center to any point on the circle is called the ***radius r***



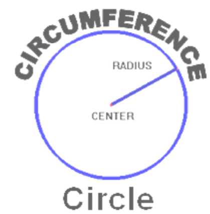

The distance between $(h, k)$ and $(x, y)$

$$\sqrt{(x-h)^2+(y-k)^2} = r$$

Squaring both sides of this equation yields:

the **standard form** of the equation of a Circle

$$(x - h)^2 + (y - k)^2 = r^2.$$




$$A = \pi r^2, \quad C = 2\pi r$$



**Ex 10** Write the standard form of the equation of the circle with center (0, 0) and radius 2.

**Sol:** Let center $(h, k) = (0, 0)$ and let radius $\boldsymbol{r} = 2$.

$(x - h)^2 + (y - k)^2 = r^2$ Standard form of a circle's equation.

$(x - 0)^2 + (y - 0)^2 = 2^2$ Substitute the given values.

$x^2 + y^2 = 4$ Simplify.

**Ex 11** Write the standard form of the equation of the circle with center (−2, 3) and radius 4.

**Sol** $(x - h)^2 + (y - k)^2 = r^2$ Standard form of a circle's equation.

$(x - (-2))^2 + (y - 3)^2 = 4^2$ Substitute the given values.

$(x + 2)^2 + (y - 3)^2 = 16$ Simplify.

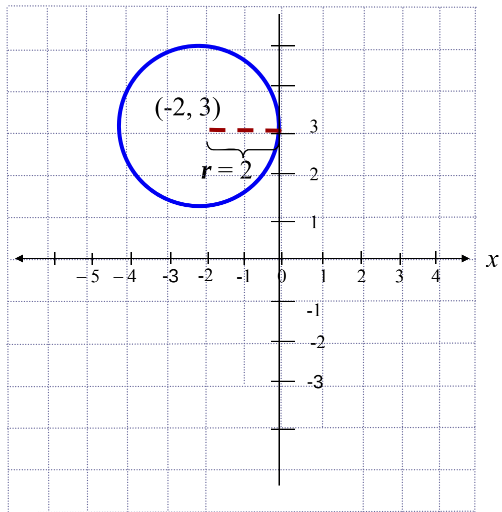
**Ex 12** Find an equation of a circle that has its center at (−2, 3), and is tangent to the y-axis. Write your answer in standard from.

**Sol:**

$$(x-h)^2 + (y-k)^2 = r^2$$

$$[x-(-2)]^2 + (y-3)^2 = 2^2$$

$$(x+2)^2 + (y-3)^2 = 4$$





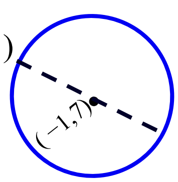

**Ex 13** Find the equation of the circle that has a diameter with end points (2,3) and (− 4,11).

**Sol** The equations is $(x-h)^2 + (y-k)^2 = r^2$

From the hypothetical adjacent figure,

the center is the midpoint of the diameter

$$(h,k) = \left(\frac{2+(-4)}{2}, \frac{3+11}{2}\right) = (-1,7)$$




the radius can be found using the distance formula as follows

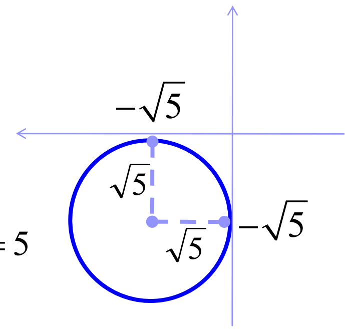
$$r = \sqrt{\left(2-(-1)^2\right)+(3-7)^2} = \sqrt{25} = 5$$

thus, the equation is $(x+1)^2 + (y-7)^2 = 25$

**Ex 14** Find the equation of the circle that is tangent to both axes, has its center in the third quadrant, and has a diameter of $\sqrt{5}$

**Sol**

the center is $\left(-\sqrt{5}, -\sqrt{5}\right)$




the equation is $\left(x+\sqrt{5}\right)^2 + \left(y+\sqrt{5}\right)^2 = 5$



**Ex 15** Determine whether each of the following equations represent circles, if yes find the center and radius of each circle:

a) $(x - 2)^2 + (y + 4)^2 = 9$

**Sol:**

$(x - 2)^2 + [y - (-4)]^2 = 3^2$

center $(h, k) = (2, -4)$ and radius $\boldsymbol{r} = 3$

$$b)\quad (x-1)^2+(y+4)^2=0$$

**Sol:**

This equation represents the point ( 1 , − 4 )

$$c)\quad (x-\frac{3}{2})^2+(y+\frac{5}{2})^2=-7$$

**Sol:**

This equation is a **contradiction**, and hence it does not represent anything



$$(x-h)^2+(y-k)^2=r^2$$

by squaring and combining like terms

$$x^2+y^2+Ax+By+C=0.$$

by completing the square on $x$ and on $y$.

$$(x-h)^2+(y-k)^2=r^2$$

d) $x^2 + y^2 + 4x - 6y - 23 = 0$

**Sol:**

$x^2 + 4x + y^2 - 6y = 23$

$[x^2 + 4x + (2)^2] + [y^2 - 6y + (-3)^2] = 23 + 4 + 9$

$(x + 2)^2 + (y - 3)^2 = 36$

$[x - (-2)]^2 + (y - 3)^2 = 6^2$

center $(h, k) = (-2, 3)$ and radius $\boldsymbol{r} = 6$

**How to write the equation of a circle in standard:**
1)Group all terms involving *x*.
2) Group all terms involving *y*
3) Move all constants to the right side.
4) Factor out the **equal** coefficients of $x^2$ and $y^2$ if they are not equal to ONE.
5) Complete the square.



e ) $2x^2 + 2y^2 - 6x + 10y = 1$

**Sol:**

$(2x^2 - 6x) + (2y^2 + 10y) = 1$

$2(x^2 - 3x) + 2(y^2 + 5y) = 1$

Watch

$2(x^2 - 3x + \left(-\frac{3}{2}\right)^2) + 2(y^2 + 5y + \left(\frac{5}{2}\right)^2) = 1 + 2(\frac{9}{4}) + 2(\frac{25}{4})$

$2(x - \frac{3}{2})^2 + 2(y + \frac{5}{2})^2 = 18$

$(x - \frac{3}{2})^2 + (y + \frac{5}{2})^2 = 9$

**How to write the equation of a circle in standard:**
1)Group all terms involving *x*.
2) Group all terms involving *y*
3) Move all constants to the right side.
4) Factor out the **equal** coefficients of $x^2$ and $y^2$ if they are not equal to ONE.
5) Complete the square.

Center $(h, k) = (3/2, -5/3)$ and Radius $\boldsymbol{r} = 3$

f ) $3x^2 + 2y^2 - 6x + 10y = 1$

**Sol:**

$$3x^2 + 2y^2 - 6x + 10y = 1$$

**Watch, that they are different**

hence, this equation does not represent a circle ?