1.5 INEQUALITIES

Objectives:

In this section, we will learn about:

- Properties of Inequalities.
- Solving Compound Inequalities.
- Solving Absolute Value Inequalities.
- The Critical Value Method.
- Solving Nonlinear Inequalities:
  - Polynomial Inequalities.
  - Rational Inequalities.

Solving inequalities حل المتباينات

To solve an inequality means to find the set of numbers that make the inequality a true statement.

- These numbers are called the ***solutions*** of the inequality and we say that they ***satisfy*** the inequality.
- The set of all solutions is called the ***solution set*** of the inequality.
- Inequalities that have the same solution set are said to be ***equivalent***.

Addition and Subtraction Properties

- If $a > b$ and c is a real number, then $a + c > b + c$, and $a - c > b - c$ have the same solution set.
- If $a < b$ and c is a real number, then $a + c < b + c$, and $a - c < b - c$ have the same solution set.

That is, adding / subtracting a real number to both sides of an inequality, gives **an equivalent inequality**.

Multiplication and Division Properties:

- If $c > 0$ the inequalities $a > b$, then $ac > bc$, and $\frac{a}{c} > \frac{b}{c}$ have the same solution set.

same

that is, multiplying / dividing both sides of an inequality by a **positive** real number , **does not change** inequality.

- If $c < 0$ the inequalities $a > b$, then $ac < bc$, and $\frac{a}{c} < \frac{b}{c}$ have the same solution set.

reversed

that is, multiplying / dividing both sides of an inequality by a **negative** real number , **changes** the inequality.

Ex 1 Solve the linear inequality $3 - 2x < 11$

Sol:

$$3 - 2x - 3 < 11 - 3$$ Subtract 3 from both sides.

$$-2x < 8$$ Simplify

$$\frac{-2x}{-2} > \frac{8}{-2}$$ Divide by − 3 and **reverse** the inequality

$$x > -4$$ The solution

$$\{x \mid x > -4\}$$ Expressed in set builder notation

Thus, the solution set is $(-4, \infty)$ Expressed in interval notation

Compound inequality المتباينات المركبة

is an inequality formed by joining two inequalities with "and" or "or."

Ex 2 Solve $x + 2 < 5$ **and** $2x - 6 > -8$.

Sol

Solve the first inequality.

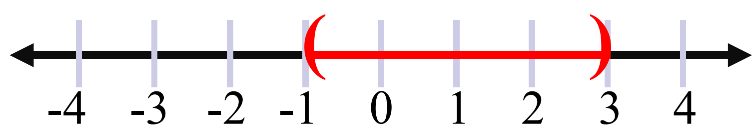
$$x + 2 < 5 \quad \text{Subtract 2.}$$
$$x < 3$$
$$\{x \mid x < 3\} \quad \text{Solution set}$$

Solve the second inequality.

$$2x - 6 > -8 \quad \text{Add 6.}$$
$$2x > -2 \quad \text{Divide by 2.}$$
$$x > -1$$
$$\{x \mid x > -1\} \quad \text{Solution set}$$

Thus, the solution set is $\{x \mid x < 3\} \cap \{x \mid x > -1\} = \{x \mid -1 < x < 3\}$

The solution set of the "and" compound inequality is the ***intersection*** of the two solution sets.



Ex 3 Solve $-3 < 2x + 1 \le 3$

Sol

$$-3 - 1 < 2x + 1 - 1 \le 3 - 1$$ Subtract 1

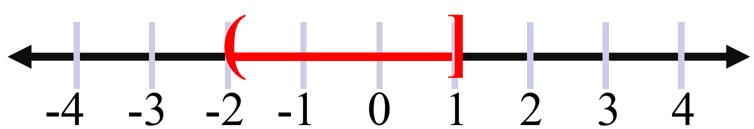
$$-4 < 2x \le 2$$ Simplify.

$$\frac{-4}{2} < \frac{2x}{2} \le \frac{2}{2}$$

$$-2 < x \le 1$$ The solution

$$\{x \mid -2 < x \le 1\}$$ Set builder notation.

Thus, the solution set is $(-2, 1]$ Interval notation



Note: The inequality $-3 < 2x + 1 \le 3$ can be written in the form $-3 < 2x + 1$ and $2x + 1 \le 3$

Ex 4 Solve $x + 5 > 6$ **or** $2x < -4$.

Sol

Solve the first inequality.

$$x + 5 > 6$$
$$x > 1$$

$\{ x \mid x > 1 \}$ Solution set

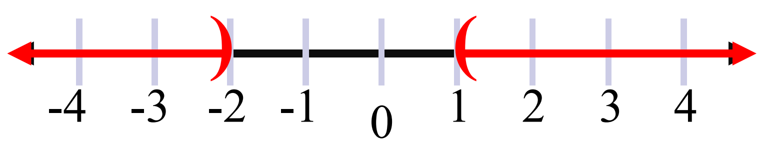
Solve the second inequality.

$$2x < -4$$
$$x < -2$$

$\{ x \mid x < -2 \}$ Solution set

Thus, the solution set is $\{x \mid x > 1\} \cup \{x \mid x < -2\}$

Since the inequalities are joined by "or" the solution set is the ***union*** of the solution sets.



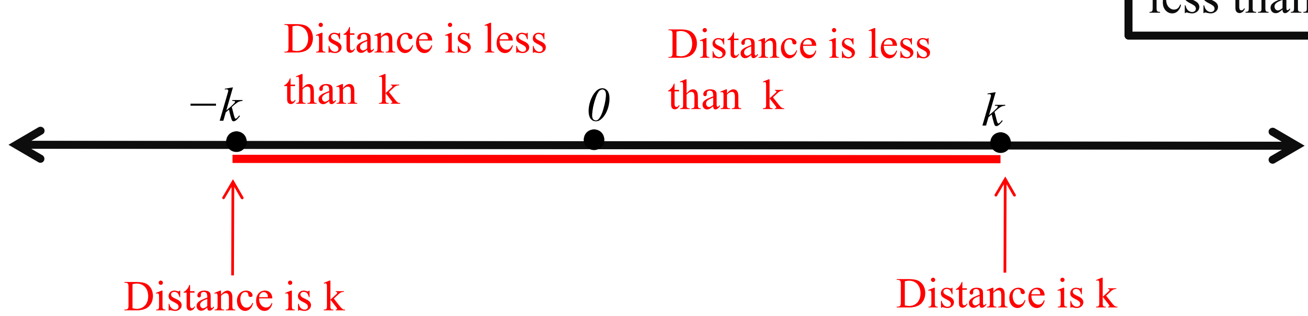
Absolute Value Inequality متباينة القيمة المطلقة

If X is an algebraic expression and k is a nonnegative number,

1. $|X| < k$ *if and only if* $-k < X < k.$

Inequality interpretation:
Distance between X and 0 is less than k

Idea:

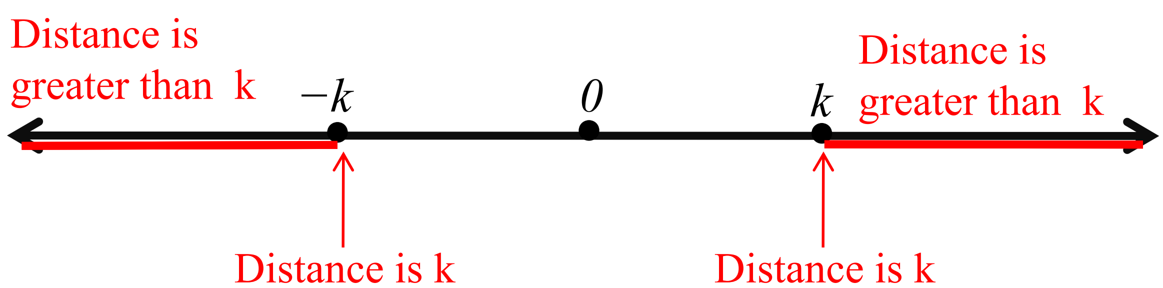


Absolute Value Inequality متباينة القيمة المطلقة

If X is an algebraic expression and k is a nonnegative number, then:

2. $|X| > k$ *if and only if* $X < -k$ or $X > k$

Inequality interpretation:
Distance between X and 0 is greater than k



Rules (1) and (2) are valid if < is replaced by ≤ and > is replaced by ≥ .

Ex 5 Solve the inequality $|x-4|<3$

Sol

$|x-4|<3$ means $-3<x-4<3$

$$-3+4<x-4+4\leq 3+4$$

$$1<x<7$$

$\{x|\ 1<x<7\}$ Set builder notation

Thus, the solution set is $(1, 7)$ Interval notation.

Apply

$|X|<k$

if and only if

$-k<X<k$

Apply

$|X| > k$

if and only if

$X < -k$ **or** $X > k$

Ex 6 Solve the inequality $|2x+3| \geq 5$

Sol $|2x+3| \geq 5$ means

$2x+3 \leq -5$ or $2x+3 \geq 5$

Solve this part separately

$2x+3-3 \leq -5-3$

$2x \leq -8$

$x \leq -4$

∪

Solve this part separately

$2x+3-3 \geq 5-3$

$2x \geq 2$

$x \geq 1$

Thus, the solution set is $\{x \mid x \leq -4 \text{ or } x \geq 1\}$ in set builder and

or the solution set is $(-\infty, -4] \cup [1, \infty)$ in interval notation

The solution of the following inequalities are straight forward:

1. $|x-5|<0$ → *No Solution*

2. $|x-5|<-2$ → *No Solution*

3. $|x-5|\leq 0$ → $x=5$

4. $|x-5|>-3$ → $(-\infty,\infty)$

5. $|x-5|>0$ → $(-\infty,5)\cup(5,\infty)$

6. $|x-5|\geq 0$ → $(-\infty,\infty)$

Ex 6 Solve the inequality $1 < |x-5| \le 2$

Sol

One way to do this inequality is rewrite it as

$$1 < |x-5| \qquad and \qquad |x-5| \le 2$$

Now solve each one **separately as follows**

$$1 < x-5 \ \ or \ \ x-5 < -1 \qquad and \qquad -2 \le x-5 \le 2$$

$$x > 6 \ \ or \ \ x < 4 \qquad and \qquad 3 \le x \le 7$$

The solution set is [3, 4) U (6, 7]

Ex 7 Solve the inequality

$$x^2 < 7x - 10$$

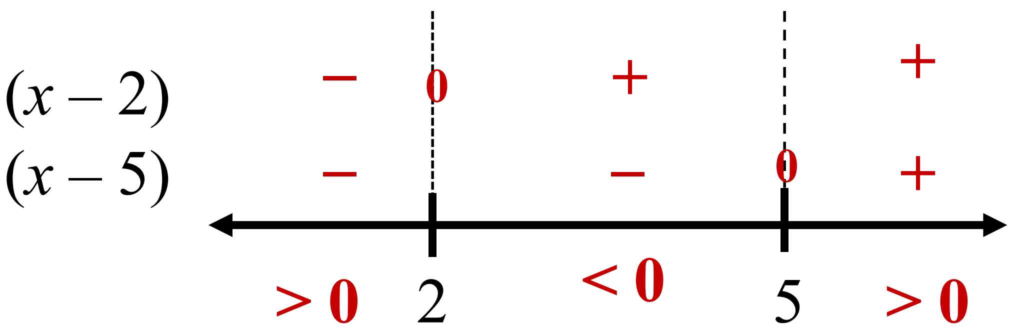
Sol

$$x^2 - 7x + 10 < 0$$

$$(x-2)(x-5) < 0$$

Common Mistake: $(x-2) < 0$, $(x-5) < 0$

Critical Values: $x = 2$ and $x = 5$



True

The solution set is the interval (2, 5).

3. Critical Value Method

Strategy used to solve **Polynomial Inequalities** and **Rational Inequalities :**

1) Express the inequality such that one side is ***zero***
2) Simplify the nonzero side by **combing fractions or like terms**
3) Factor all polynomial(s), numerator and denominator, **if possible**.
4) Find the **real** zeros(**critical values)** of the of the numerator and denominator.
5) Locate these critical values on a number line, thereby dividing the number line into ***test intervals.***
5) Choose one representative number within each test interval and decide the sign of each factor.
6) Determine the sign of the entire expression.
7) Write the solution set, selecting the interval(s) that produced a true statement.

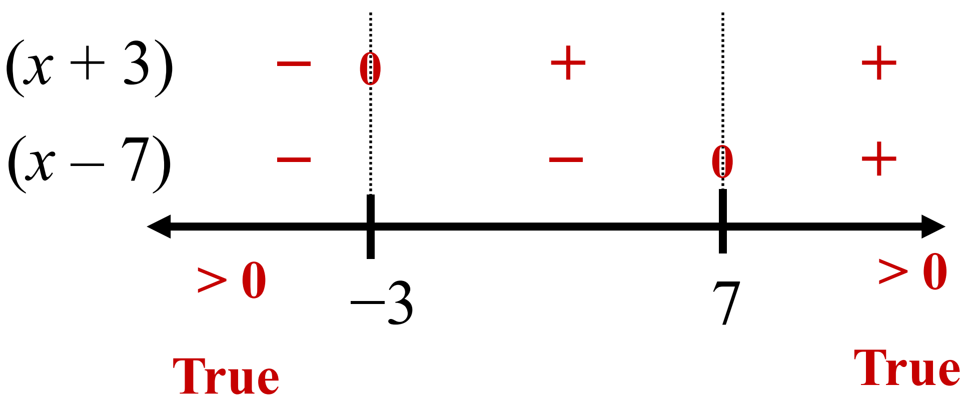
Note: **Don't multiply / divide an inequality by an algebraic expression**

Ex 8 Solve the inequality $\dfrac{x+3}{x-7} > 0$

Sol

Common Mistake: $(x+3) > 0 \;,\; (x-7) > 0$

Critical Values: $x = -3$ and $x = 7$



The solution set is the interval **$(-\infty, -3) \cup (7, \infty)$.**

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Ex 9 Solve the inequality $\dfrac{x+1}{x+3} \le 2$

Sol

$$\frac{x+1}{x+3} - 2 \le 0$$

$$\frac{x+1-2(x+3)}{x+3} \le 0$$

$$\frac{-x-5}{x+3} \le 0$$

Common Mistake:
$(x+1) \le 2(x+3)$

Critical values: $x = -5$ and $x = -3$

3. Critical Value Method

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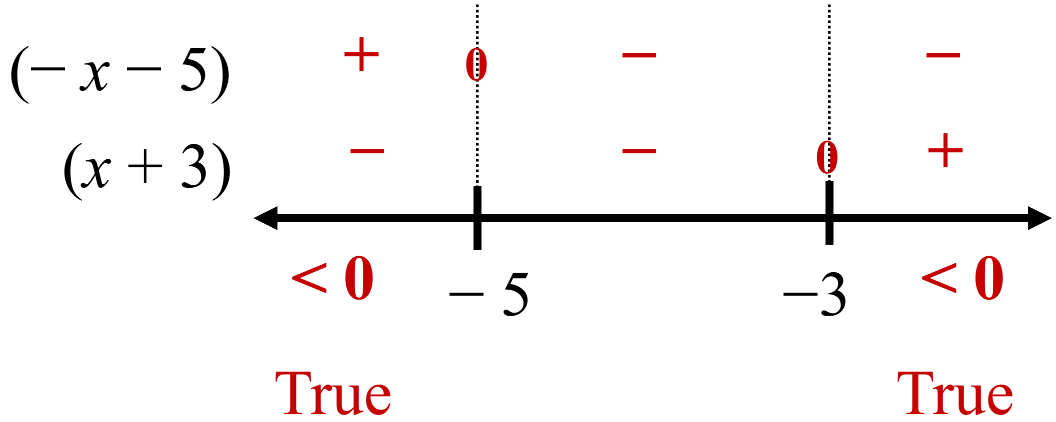


The solution set is the interval $(-\infty, -5] \cup (-3, \infty)$

Ex 10 Find the set of all real numbers k such that the equation $3x^2 - 2(k+1)x + 3 = 0$ has two non real solutions.

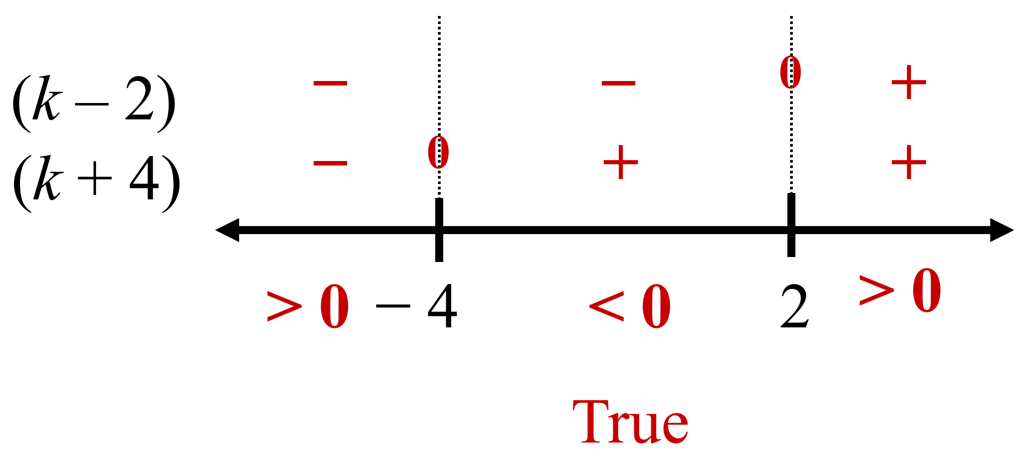
Sol This equation has non real solutions if $D = b^2 - 4ac \overset{watch}{<} 0$

$$a = 3,\ b = -2(k+1),\ c = 3$$

$$\Rightarrow \left(-2(k+1)\right)^2 - 4(3)(3) \overset{watch}{<} 0$$

$$\Rightarrow \quad 4k^2 + 8k - 32 < 0 \qquad or \quad k^2 + 2k - 8 < 0$$

$$or \quad (k+4)(k-2) < 0$$



, thus $-4 < k < 2$

Ex 11 Solve the inequality

$$\frac{1}{x^2+x-6} \le \frac{3}{x+3}$$

Sol

$$\frac{1}{(x+3)(x-2)} - \frac{3}{x+3} \le 0$$

$$\frac{1-3(x-2)}{(x+3)(x-2)} \le 0$$

$$\frac{-3x+7}{(x+3)(x-2)} \le 0$$

x−2	−		−	0	+		+
x+3	−	0	+		+		+
−3x+7	+		+		+	0	−
		−3		2		7/ 3	
	>0		< 0 **True**		> 0		< 0 **True**

The solution set is $(-3, 2) \cup [7/3, \infty)$

Common mistake:

$$x+3 \le 3\left(x^2+2x-3\right)$$

3. Critical Value Method

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Ex 12 Solve the inequality

Sol
$$\frac{-(x-3)^{10}(x-5)^{5}}{(x+2)(x^2+4)} \le 0$$

The critical valuea are 3, 5 and −2

Notice also that $x^2 + 4$ has no real zeros and it is always positive valued.

		−2		3		5	
$-$	−		−		−		−
x^2+4	+		+		+		+
$x+2$	−	0	+		+		+
$(x-5)^5$	−		−		−	0	+
$(x-3)^{10}$	+		+	0	+		+
	< 0 **True**		> 0		> 0		**< 0** **True**

3. Critical Value Method

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$$\text{the S.S.} = (-\infty,-2) \cup \{3\} \cup [5,\infty)$$

Ex 13 Solve the inequality

$$x^3 + 7x^2 \le -12x$$

Sol

$$x^3 + 7x^2 + 12x \le 0$$

$$x\left(x^2 + 7x + 12\right) \le 0$$

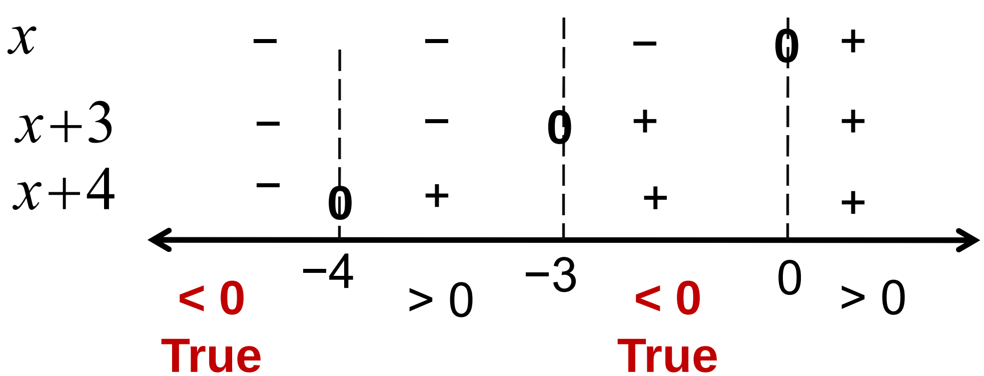
$$x(x+3)(x+4) \le 0$$

Common Mistake 1:
Divide by x

Common Mistake 2:
$x \le 0,\ x+3 \le 0,\ x+4 \le 0$

Critical values: $-4, -3$ and 0

$$\text{the S.S.} = (-\infty, -4] \cup [-3, 0]$$



3. Critical Value Method
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