1.4 OTHER TYPES OF EQUATIONS

Objectives:

In this section, we will learn about how to solve other types of equations such as:

- Polynomial Equations.
- Rational Equations.
- Radical and Rational Exponent Equations.
- Equations that are Quadratic in Form.

1. Polynomial Equations

Some polynomial equations that are **neither linear nor quadratic** can be solved by factoring**.**

Ex 1 Solve by factoring: $x^3 = -1$

Sol: $x^3 + 1 = 0$

$$(x+1)(x^2 - x + 1) = 0$$

$$x+1=0 \quad or \quad x^2 - x + 1 = 0$$

Strategy to solve polynomial equations:

a) If necessary, rewrite the equation in the form
$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + a_0 = 0$$

b) Factor , preferably, till you get product of linear or/and quadratic factors.

c) Apply the ***zero-product principle***, setting each factor equal to zero.

d) Solve the equations in step c

$$(x+1)\left(x^2-x+1\right)=0$$

$$x=-1 \qquad x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2}$$

$$x=\frac{1\pm\sqrt{3}\ i}{2}$$

Checking the proposed results confirms that the solution set is

$$\left\{-1,\ \frac{1\pm\sqrt{3}\ i}{2}\right\}$$

Ex 2 Solve by factoring: $3x^4 = 27x^2$

Sol:

$$3x^4 - 27x^2 = 0$$

$$3x^2(x^2 - 9) = 0$$

Strategy to solve polynomial equations:

a) If necessary, rewrite the equation in the form

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + a_0 = 0$$

b) Factor , preferably, till you get product of linear or/and quadratic factors.

c) Apply the ***zero-product principle***, setting each factor equal to zero.

d) Solve the equations in step c

$$3x^2 = 0 \quad \text{or} \quad x^2 - 9 = 0$$

$$x^2 = 0 \qquad\qquad x^2 = 9$$

$$x = \pm\sqrt{0} \qquad\qquad x = \pm\sqrt{9}$$

$$x = 0\,(double \quad solution) \qquad\qquad x = \pm 3$$

Checking the proposed results confirms that the solution set is $\{-3, \quad 0, \quad 3\}$.

Strategy to solve polynomial equations:

a) If necessary, rewrite the equation in the form

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + a_0 = 0$$

b) Factor , preferably, till you get product of linear or/and quadratic factors.

c) Apply the ***zero-product principle***, setting each factor equal to zero.

d) Solve the equations in step c

Ex 3 Solve by factoring: $x^3 + x^2 = 4x + 4$

Sol : $x^3 + x^2 - 4x - 4 = 0$

$$(x^3 + x^2) + (-4x - 4) = 0$$

$$x^2(x + 1) - 4\,(x + 1) = 0$$

$$(x + 1)\,(x^2 - 4) = 0$$

$(x + 1) = 0$ or $(x^2 - 4) = 0$ — Apply the zero product principle.

$x = -1$ — $x^2 = 4$ — Solve the resulting equations

$x = \pm 2$

Checking the proposed results confirms the solution set is $\{-2, \ -1, \ 2\}$.

2. Rational Equations

A rational equation is an equation that involves **fractions** in which the numerators and/or the denominators of the fractions are polynomials.

Such as: $$\frac{x}{x-3} = \frac{9}{x-3} - 5$$

Ex 4 Solve the equation

$$\frac{1}{x-4}-\frac{5}{x+2}=\frac{6}{x^2-2x-8}$$

Sol :

$$\frac{1}{x-4}-\frac{5}{x+2}=\frac{6}{(x-4)(x+2)}$$

LCD: $(x-4)(x+2)$; $x \neq 4, -2$

Strategy to solve Rational Equations:

a) Remove all the all denominators by multiplying both sides by the LCD of all denominators

b) Solve the resultant equation

c) Check the proposed solutions in the original equation.

$$(x-4)(x+2)\left(\frac{1}{x-4}-\frac{5}{x+2}\right)=(x-4)(x+2)\cdot\frac{6}{(x-4)(x+2)} \quad , x \neq 4, -2$$

$$\cancel{(x-4)}(x+2)\frac{1}{\cancel{x-4}}-(x-4)\cancel{(x+2)}\frac{5}{\cancel{x+2}}=\cancel{(x-4)}\cancel{(x+2)}\cdot\frac{6}{\cancel{(x-4)}\cancel{(x+2)}}$$

$, x \neq 4, -2$

$$(x-4)(x+2)\frac{1}{x-4}-(x-4)(x+2)\frac{5}{x+2}=(x-4)(x+2)\cdot\frac{6}{(x-4)(x+2)}$$

$$x + 2 - 5x + 20 = 6 \qquad , x \neq 4, -2$$

$$x = 4, \qquad , x \neq 4, -2$$

Checking the proposed solution in the original equation , shows that 4 is **not** a solution . The solution set is the **empty set**.

Strategy to solve Rational Equations:

a) Remove all the all denominators by multiplying both sides by the LCD of all denominators

b) Solve the resultant equation

c) Check the proposed solutions in the original equation.

Ex 5 Solve the equation: $\frac{8}{2m+1}-\frac{1}{m-2}=\frac{5}{2m+1}$

Sol :

LCD: $(2m+1)(m-2)$; $m \neq -1/2,\ 2$

$$(2m+1)(m-2)\left(\frac{8}{2m+1}-\frac{1}{m-2}\right)$$

$$=(2m+1)(m-2)\cdot\frac{5}{2m+1}$$

$$\cancel{(2m+1)}(m-2)\cdot\frac{8}{\cancel{2m+1}}-(2m+1)\cancel{(m-2)}\cdot\frac{1}{\cancel{m-2}}$$

$$=\cancel{(2m+1)}(m-2)\cdot\frac{5}{\cancel{2m+1}}$$, $m \neq -1/2,\ 2$

Strategy to solve Rational Equations:

a) Remove all the all denominators by multiplying both sides by the LCD of all denominators

b) Solve the resultant equation

c) Check the proposed solutions in the original equation..

$$(2m+1)(m-2)\cdot\frac{8}{2m+1}-(2m+1)(m-2)\cdot\frac{1}{m-2}$$

$$=(2m+1)(m-2)\cdot\frac{5}{2m+1}$$

$$8m-16-2m-1=5m-10 \quad ,\ m\neq -1/2,\ 2$$

$$m=7,$$

Strategy to solve Rational Equations:

a) Remove all the all denominators by multiplying both sides by the LCD of all denominators

b) Solve the resultant equation

c) Check the proposed solutions in the original equation..

Checking the proposed solution in the original equation , shows that 7 is a solution . The solution set is **{ 7 }.**

3. Radical Equations

are equations containing **variables within radical** signs.

such as: $\sqrt{x+9}+\sqrt{x}=7$, $\sqrt{x-3}=3$, $\sqrt[3]{x+1}+2=3$

The Power Principle:

If P and Q are algebraic expressions and n is a positive integer, then every solution of $P=Q$ is a solution of $P^n=Q^n$.

Ex 6 Solve: $x+\sqrt{26-11x}=4$

Sol:

$$\sqrt{26-11x}=4-x$$

$$\left(\sqrt{26-11x}\right)^2=(4-x)^2$$

$$26-11x=16-8x+x^2$$

$$x^2+3x-10=0$$

$$(x+5)(x-2)=0$$

$$x=-5 \quad \text{or} \quad x=2$$

To solve a radical equation containing one square root:

1. Isolate the radical on one side of the equation.
2. Square both sides of the equation.
3. Simplify the **new resulting** equation using the **perfect square rule** and solve for the variable.
4. Check the proposed solutions in the original equation.

Checking the proposed solutions in the original equation, shows that

$$(2)+\sqrt{26-11(2)}=4 \qquad (-5)+\sqrt{26-11(-5)}=4$$

Thus both − 5 and 2 are solution s. The solution set is **{ − 5 , 2 }.**

Ex 7 Solve $x+2\sqrt{x+2}=6$

Sol:

$$2\sqrt{x+2}=6-x$$

$$4(x+2)=36-12x+x^2$$

$$x^2-16x+28=0$$

$$(x-2)(x-14)=0$$

$$x=2 \text{ or } x=14$$

To solve a radical equation containing one square root:

1. Isolate the radical on one side of the equation.
2. Square both sides of the equation.
3. Simplify the **new resulting** equation using the **perfect square rule** and solve for the variable.
4. Check the proposed solutions in the original equation.

Checking the proposed solutions in the original equation, shows that

$$(2)+2\sqrt{(2)+2}=6 \qquad (14)+2\sqrt{(14)+2}\neq 6$$

Thus, the solution set is $\{\,2\,\}$.

Ex 8 Find all real solutions of the equation

$$2\cdot\sqrt[3]{3x}+3=x$$

Sol:

$$2\cdot\sqrt[3]{3x}+3=x$$

$$2\cdot\sqrt[3]{3x}=x-3$$

$$\left(2\sqrt[3]{3x}\right)^3=(x-3)^3$$

$$24x=x^3-9x^2+27x-27$$

$$x^3-9x^2+3x-27=0$$

$$x^2(x-9)+3(x-9)=0$$

$$(x^2+3)(x-9)=0$$

$$x^2+3=0 \quad \text{or } x-9=0$$

$$x=\pm\sqrt{3}\ i \qquad \textbf{or } x=9$$

To solve a radical equation containing one cube root:

1. Isolate the radical on one side of the equation.
2. Cube both sides of the equation.
3. Simplify the **new resulting** equation using the **perfect cube rule** and solve for the variable.
4. Check the proposed solutions in the original equation.

Checking the proposed solution in the original equation, shows that 9 is a solution.

$$2\cdot\sqrt[3]{3(9)}+3=(9)$$

Extraneous Solution is any solution of $P^n=Q^n$ that is not a solution of $P=Q$. *These solutions may be introduced whenever we raise each side of an equation to* ***an even*** *power.*

Ex 9 Solve $\sqrt{3x+1}-\sqrt{x+4}=1$

Sol:

$$\sqrt{3x+1}=\sqrt{x+4}+1$$

$$\left(\sqrt{3x+1}\right)^2=\left(\sqrt{x+4}+1\right)^2$$

$$3x+1=x+4+2\sqrt{x+4}+1$$

$$2x-4=2\sqrt{x+4}$$

$$(2x-4)^2=\left(2\sqrt{x+4}\right)^2$$

$$4x^2-16x+16=4(x+4)$$

$$4x^2-20x=0$$

$$4x(x-5)=0$$

$$x=0 \quad \text{or} \quad x=5$$

To solve a radical equation containing two square roots:

1. Isolate one radical on one side of the equation.
2. Square both side of the equation and simplify.
3. Isolate the other radical.
4. Square both sides again and simplify.
5. Solve the resulting new equation.
6. Check the proposed solutions in the original equation.

Checking the proposed solution in the original equation, shows that **5 is a solution** and **0 is not a solution** (extraneous solution).
Thus, the solution set is { 5 }

$$\sqrt{3(5)+1}-\sqrt{(5)+4}=1$$

$$\sqrt{3(0)+1}-\sqrt{(0)+4}\neq 1$$

4. Solving Radical Equations of the form $x^{m/n} = k$

Ex 10 Solve $3x^{3/4} - 6 = 0$

Sol:

$$3x^{3/4} = 6$$

$$\frac{3x^{3/4}}{3} = \frac{6}{3}$$

$$x^{3/4} = 2$$

$$(x^{3/4})^{4/3} = 2^{4/3}$$

$$x = 2^{4/3}$$

Checking in the original equation confirms that the solution set is $\{\ 2^{4/3}\ \}$

Solving Radical Equations of the form $x^{m/n} = k$

Assume that *m* and *n* are positive integers, *m/n* is in lowest terms, and *k is* a real number.

1) Isolate the expression with the rational exponent.
2) Raise both sides of the equation to the *n/m* power.

If *m* is even:	**If *m* is odd:**
$x^{m/n} = k$	$x^{m/n} = k$
$(x^{m/n})^{n/m} = \pm k^{n/m}$	$(x^{m/n})^{n/m} = k^{n/m}$
$x = \pm k^{n/m}$	$x = k^{n/m}$

(Note: An odd index has only one root to consider).

3) Check the proposed solutions in the original equation.

Ex 11 Solve $x^{2/3} - \frac{3}{4} = -\frac{1}{2}$

Sol:

$$x^{2/3} = \frac{1}{4}$$

$$\left(x^{2/3}\right)^{3/2} = \pm\left(\tfrac{1}{4}\right)^{3/2}$$

$$x = \pm\frac{1}{8}$$

Checking in the original equation confirms the solution set $\left\{-\frac{1}{8}, \frac{1}{8}\right\}$.

Solving Radical Equations of the form $x^{m/n} = k$

Assume that *m* and *n* are positive integers, *m/n* is in lowest terms, and *k is* a real number.

1) Isolate the expression with the rational exponent.
2) Raise both sides of the equation to the *n/m* power.

If *m* is even:	**If *m* is odd:**
$x^{m/n} = k$	$x^{m/n} = k$
$(x^{m/n})^{n/m} = \pm k^{n/m}$	$(x^{m/n})^{n/m} = k^{n/m}$
$x = \pm k^{n/m}$	$x = k^{n/m}$

(Note: An odd index has only one root to consider).

3) Check the proposed solutions in the original equation.

5. Equations that are Quadratic in form

These are equations which can be solved by transforming it to a quadratic equation.

Ex 12 Solve $x^4 - 8x^2 - 9 = 0$

Sol: $(x^2)^2 - 8x^2 - 9 = 0$

Let $u = x^2$

$u^2 - 8u - 9 = 0$

$(u - 9)(u + 1) = 0$

$u - 9 = 0$ or $u + 1 = 0$

$u = 9$ $\quad$ $u = -1$

$x^2 = 9$ $\quad$ $x^2 = -1$

$x = \pm\sqrt{9}$ $\quad$ $x = \pm\sqrt{-1}$

$x = \pm 3$ $\quad$ $x = \pm i$

The solution set is $\{-3, 3, -i, i\}$.

Strategy to solve equations that are quadratic in form

1) Rewrite the equation in form

$$a(\square)^2 + b(\square) + c = 0$$

where $\square$ is an algebraic expression

2) Use the transformation $u = \square$ to get a quadratic equation.
3) Solve the new resulting quadratic equation for u.
4) Substitute back to get the value(s) of the original variable.
5) <u>Check</u> the proposed solutions in the <u>original equation</u>

Ex 13 Solve $5x^{2/3} + 11x^{1/3} + 2 = 0$

Sol:

$$5(x^{1/3})^2 + 11\,x^{1/3} + 2 = 0$$

Let $t = x^{1/3}$

$$5t^2 + 11t + 2 = 0$$

$$(5t + 1)(t + 2) = 0$$

$t = -1/5$ or $t = -2$

$x^{1/3} = -1/5$ or $x^{1/3} = -2$

$(x^{1/3})^3 = (-1/5)^3$ or $(x^{1/3})^3 = (-2)^3$

$x = -1/125$ or $x = -8$

Checking in the original equation confirms

that the solution set $\left\{-\frac{1}{125},\ -8\right\}$.

Strategy to solve equations that are quadratic in form

1) Rewrite the equation in form

$$a\left(\square\right)^2 + b\left(\square\right) + c = 0$$

where $\square$ is an algebraic expression

2) Use the transformation $u = \square$ to get a quadratic equation.
3) Solve the new resulting quadratic equation for u.
4) Substitute back to get the value(s) of the original variable.
5) Check the proposed solutions in the original equation

Ex 14 Solve $2x^{1/2} + 5x^{1/4} - 3 = 0$

Sol:

$$2\left(x^{1/4}\right)^2 + 5x^{1/4} - 3 = 0$$

Let $t = x^{1/4}$

$$2t^2 + 5t - 3 = 0$$

$$(2t - 1)(t + 3) = 0$$

$t = 1/2$ or $t = -3$

$x^{1/4} = 1/2$ or $x^{1/4} = -3$

(Rejected ?)

$$(x^{1/4})^4 = (1/2)^4$$

$$x = 1/16$$

Checking in the original equation confirms

that the solution set $\{1/16\}$.

Strategy to solve equations that are quadratic in form

1) Rewrite the equation in form

$$a(\square)^2 + b(\square) + c = 0$$

where $\square$ is an algebraic expression

2) Use the transformation $u = \square$ to get a quadratic equation.
3) Solve the new resulting quadratic equation for u.
4) Substitute back to get the value(s) of the original variable.
5) Check the proposed solutions in the original equation