1.3 QUADRATIC EQUATIONS

Objectives:

In this section, we learn about:

- Solving Quadratic Equations by Factoring
- Solving Quadratic Equations by Taking Square Roots
- Solving Quadratic Equations by Completing the Square
- Solving Quadratic Equations by Using the Quadratic Formula
- The Discriminant of a Quadratic Equation
- Application of Quadratic Equation

Definition of a Quadratic Equation المعادلة التربيعية

A **quadratic equation** in x is an equation that can be written in the

standard form

$$ax^2 + bx + c = 0$$

where a, b, and c are real numbers with $a \neq 0$.

A quadratic equation in x is also called a ***second-degree polynomial equation*** in x.

The Zero Product Principle

If the product of two algebraic expressions is zero, then at least one of the factors is equal to zero.

If $A \cdot B = 0$, then $A = 0$ or $B = 0$.

product zero

Quadratic equations can be solved by:

1. Factoring
2. Taking square roots of both sides
3. Completing the square
4. Using the quadratic formula

1. Solving a Quadratic Equation by Factoring

a) If necessary, rewrite the equation in the form $ax^2 + bx + c = 0$, moving all terms to one side, thereby ***obtaining zero*** on the other side.

b) Factor.

c) Apply the ***zero-product principle***, setting each factor equal to zero.

d) Solve the equations in step c.

Ex 1 Factor and Solve the quadratic equation: $x^2 - 7x + 10 = 0$

Sol:

$$(x-2)(x-5)=0$$

$$x-2=0 \quad \text{or} \quad x-5=0$$

$$x=2 \qquad\qquad x=5$$

1. Solving a Quadratic Equation by Factoring

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Ex 2 Solve the quadratic equation $4x^2 = 2x$

Sol:

$$4x^2 - 2x = 0$$

$$2x(2x - 1) = 0$$

$$2x = 0 \quad \text{or} \quad 2x - 1 = 0$$

$$x = 0 \qquad x = 1/2$$

A check shows that the solution set for the given equation is $\{0, 1/2\}$.

1. Solving a Quadratic Equation by Factoring

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Ex 3 Solve the quadratic equation $(x-1)(x+3)=5$

Sol:

$$(x-1)(x+3)=5$$ ← **Not zero**

Notice that the zero product principle **dose not** apply here

$$x^2+2x-8=0$$

$$(x+4)(x-2)=0$$

$$x+4=0 \quad \text{or} \quad x-2=0$$

$$x=-4 \qquad \text{or} \qquad x=2$$

A check shows that the solution set for the given equation is $\{-4, 2\}$

1. Solving a Quadratic Equation by Factoring

a) If necessary, rewrite the equation in the form $ax^2+bx+c=0$, moving all terms to one side, thereby obtaining zero on the other side.

b) Factor.

c) Apply the ***zero-product principle***, setting each factor equal to zero.

d) Solve the equations in step c.

2. The Square Root Method:

If x is an algebraic expression and c is a positive real number, then $x^2 = c$ has exactly two solutions $\pm\sqrt{c}$

Equivalently, If $x^2 = c$, then $x = \pm\sqrt{c}$.

Ex 4 Solve by the square root method:

$a.\ 4x^2 = 20$ $b.\ (x-2)^2 + 6 = 0$

Sol: *a)* $x^2 = 5$ Divide both sides by 4

$x = \pm\sqrt{5}$ Apply the square root method

The solution set is $\{-\sqrt{5}, \sqrt{5}\}$.

Sol: ***b)***

$$(x-2)^2+6=0$$

$$(x-2)^2=-6$$

$$x-2=\pm\sqrt{-6}$$

$$x=2\pm\sqrt{6}\,i$$

The solution set is $\left\{2-\sqrt{6}\,i,\ \ 2+\sqrt{6}\,i\right\}$.

2. The Square Root Method:

1. If necessary write the equation in the form $x^2 = c$
2. Apply the square root property

Or equivalently,

If $x^2 = c$, then $x = \pm\sqrt{c}$.

3. Completing the square اكمال المربع

If $x^2 + bx$ is a binomial, then by adding $\left(\frac{b}{2}\right)^2$, a perfect square trinomial will result. That is,

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2.$$

Note that $(b/2)^2$ is the square of half the coefficient of x

Expression	Add	Complete the Square
$x^2 - 12x$	$\left(\frac{-12}{2}\right)^2 = 36$	$x^2 - 12x + 36 = (x - 6)^2$
$x^2 + \sqrt{3}\,x$	$\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$	$x^2 + \sqrt{3}\,x + \frac{3}{4} = \left(x + \frac{\sqrt{3}}{2}\right)^2$

Ex 5 Solve by completing the square: $x^2 - 6x + 2 = 0$

Sol: $x^2 - 6x + 2 = 0$ Notice that this equation cannot be factored?

$x^2 - 6x = -2$ Move the constant to the R.H.S.

Add $\left(\frac{b}{2}\right)^2$ to both sides $(-6/2)^2 = (-3)^2 = 9$ **(to complete the square)**

$$x^2 - 6x + (-3)^2 = -2 + 9$$

$$x^2 - 2 \cdot x \cdot 3 + (-3)^2 = -2 + 9$$

Notice that the terms in the left side form a perfect square

First square (x^2); Twice First times second ($2 \cdot x \cdot 3$); last square ($(-3)^2$)

$$(x-3)^2 = 7$$

$$(x-3)^2 = 7$$

$$x-3=\pm\sqrt{7} \qquad x = 3\pm\sqrt{7}$$

The solution set is $\left\{3-\sqrt{7},\ 3+\sqrt{7}\right\}$.

Ex 6 Solve by completing the square $3x^2 - 2x + 4 = 0$

Sol:

$$3x^2 - 2x + 4 = 0$$

This equation cannot be factored ?

$$\frac{3x^2}{3} - \frac{2x}{3} + \frac{4}{3} = \frac{0}{3}$$

Divide both sides by 3

$$x^2 - \frac{2}{3}x + \frac{4}{3} = 0$$

$$x^2-\frac{2}{3}x+\frac{4}{3}=0$$

$$x^2-\frac{2}{3}x=-\frac{4}{3}$$

Move the constant to the R.H.S.

$$x^2-\frac{2}{3}x+\left(-\frac{1}{3}\right)^2=-\frac{4}{3}+\frac{1}{9}$$

Complete the square

$$x^2-2\cdot\frac{1}{3}\cdot x+\left(-\frac{1}{3}\right)^2=-\frac{4}{3}+\frac{1}{9}$$

First square — Twice First times second — last square

Notice that the terms in the left side form a perfect square

$$\left(x-\frac{1}{3}\right)^2=-\frac{11}{9} \qquad \Rightarrow x-\frac{1}{3}=\pm\sqrt{-\frac{11}{9}}$$

Apply the square root property

$$\Rightarrow x=\frac{1}{3}\pm\frac{\sqrt{11}}{3}i$$

The solution set is $\left\{\frac{1}{3}+\frac{\sqrt{11}}{3}i,\ \frac{1}{3}-\frac{\sqrt{11}}{3}i\right\}$

4. The Quadratic Formula

The solutions of a quadratic equation in **standard form**

$$ax^2 + b\,x + c = 0,$$

with $a \neq 0$, are given by the ***quadratic formula***

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Ex 7 Solve the equation $3x^2 = 2x - 4$ using the quadratic formula.

Sol:

$$3x^2 - 2x + 4 = 0$$ Rewrite the equation in standard from

The coefficient are $a = 3,\ b = -2,\ c = 4.$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(4)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 - 48}}{6}$$ Simplify

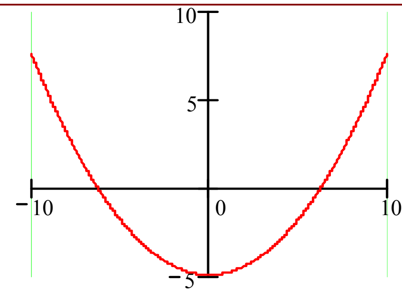
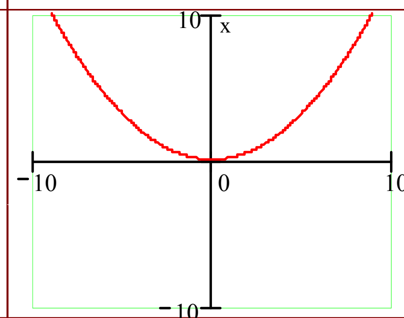
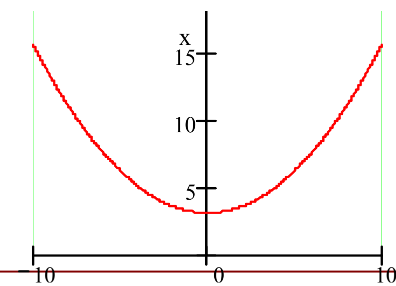
$$= \frac{2 \pm \sqrt{-44}}{6}$$ Simplify

$$= \frac{2 \pm 2\sqrt{11}\, i}{6}$$ Principle square root of a negative real number

$$= \frac{\cancel{2}(1 \pm \sqrt{11}\, i)}{\cancel{2} \cdot 3}$$ Simplify

$$x = \frac{1}{3} \pm \frac{\sqrt{11}}{3}\, i$$ Standard form of a complex number.

The solutions are: $\frac{1}{3} \pm \frac{\sqrt{11}}{3}\, i$

Discriminant **$D = b^2 - 4ac$**	**Kinds of Solutions to** **$ax^2+bx+c = 0$**	**Graph of** **$Y = ax^2+bx+c$**
$D > 0$	2 distinct real solutions	
$D = 0$	1 real solution (a double solution)	
$D < 0$	N0 real solutions 2 complex solutions	

Ex 8 a) State the number of solutions of the following equation without solving the equation: $x^2 + 3x + 3 = 0$

Sol:

$$D = b^2 - 4ac \qquad a = 1,\ b = 3,\ c = 3.$$

$$D = 9 - 4(1)(3)$$

$$D = 9 - 12 = -3 < 0$$

The equation has two complex conjugate solutions

b) Find all values of K such that the equation $4x^2 + Kx + 25 = 0$ has exactly one solution(a double solution).

Sol:

In this example $a = 4,\ b = K,\ c = 25$

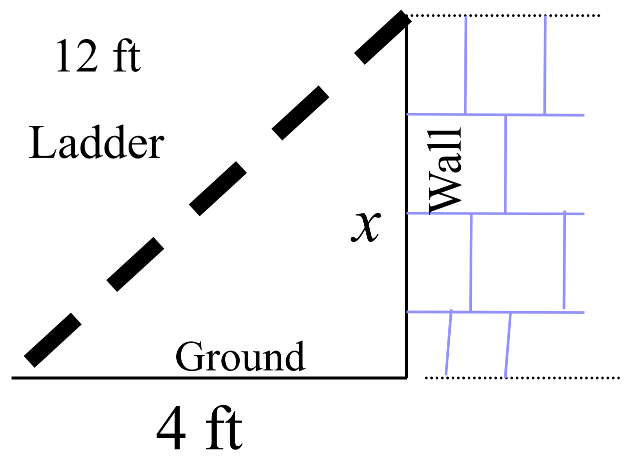
$$\Rightarrow D = b^2 - 4ac = \mathbf{0}$$

$$\Rightarrow K^2 - 4 \cdot 4 \cdot 25 = 0 \qquad \Rightarrow K^2 - 400 = 0$$

$$\Rightarrow K = \pm 20$$

Ex 9 A 12-foot ladder is leaning against a building. How high on the building will the ladder reach when the bottom of the ladder is 4 feet from the building?

Sol:



$$12^2 = 4^2 + x^2$$

$$144 - 16 = x^2$$

$$x^2 = 128$$

$$x = \pm\sqrt{128}$$

Negative sign is rejected ?

$$x = 11.3 \text{ foot}$$

Sum and Product of the roots Theorem

Let $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$, be a quadratic equation. Then x_1 and x_2 are roots of the equation

if and only if

$$x_1 + x_2 = \frac{-b}{a} \quad and \quad x_1.x_2 = \frac{c}{a}$$

Ex 10 If the Sum and product of the roots of $x^2 + 3x + 3 = 0$ are A and B respectively. Find A+B

Sol:

$a = 1, b = 3, c = 3.$

$$x_1 + x_2 = \frac{-b}{a} = -3 \quad \Rightarrow A = -3$$

$$\Rightarrow A + B = 0$$

and $x_1.x_2 = \frac{c}{a} = 3 \quad \Rightarrow B = 3$

Ex 11 If 3 is a solution of the equation $Kx^2+3x+9=0$, then find :

a) the value of *K*.

b) the other solution.

Sol:

Since x = 3 is a solution, then it satisfies the equation, i.e.

$K(3)^2+3(3)+9=0$, *thus* $K=-2$

The equation becomes $-2x^2+3x+9=0$

To find the other solution, one way is to use $x_1.x_2=\dfrac{c}{a}$

where we have $(3).x_2=\dfrac{9}{-2}$

$\Rightarrow$ the other solution is $x_2=\dfrac{-3}{2}$

Ex 12 If <u>*m* and *n* are the solutions</u> of the equation $2x^2 - 2x + 1 = 0$, then find the equation whose solutions are $3m$ and $3n$.

Sol:

Since m and n is a solution, then $m + n = - b/a$ and $mn = c/a$, i.e.

that is, $m+n = 1$ *and* $mn = 1/2$ (*)

the equation whose solutions are $3m$ and $3n$ *is* $(x - 3m)(x - 3n) = 0$

$$x^2 - 3(m+n)x + 9mn = 0$$ Writing the equation ins tandard from

$$x^2 - 3(1) + 9\left(\frac{1}{2}\right) = 0$$ Using equations in *

the equations whose solutions are 3m and 3n is

$$x^2 - 3(1) + 9\left(\frac{1}{2}\right) = 0$$