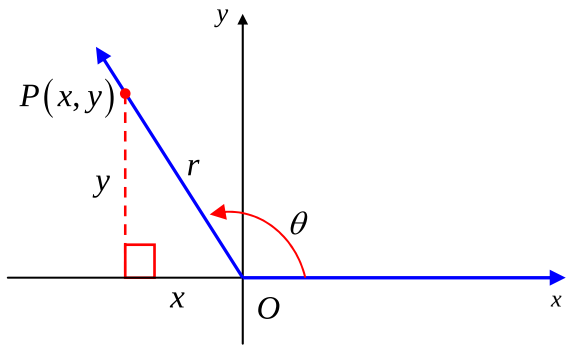
5.3: TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

I. Introduction:

In this section, we extend the definition of trigonometric functions of acute angles (*section* 5.2) to **any angle** according to the following definitions.

Let $P(x, y)$ be any point, except the origin, on the terminal side of an angle θ in standard position. Let $r = d(O, P)$, the distance from the origin to *P*. The six trigonometric functions of θ are:



1. $\sin\theta = \frac{y}{r}$
2. $\cos\theta = \frac{x}{r}$
3. $\tan\theta = \frac{y}{x},\ x \neq 0$
4. $\csc\theta = \frac{r}{y}, y \neq 0$
5. $\sec\theta = \frac{r}{x}, x \neq 0$
6. $\cot\theta = \frac{x}{y}, y \neq 0$

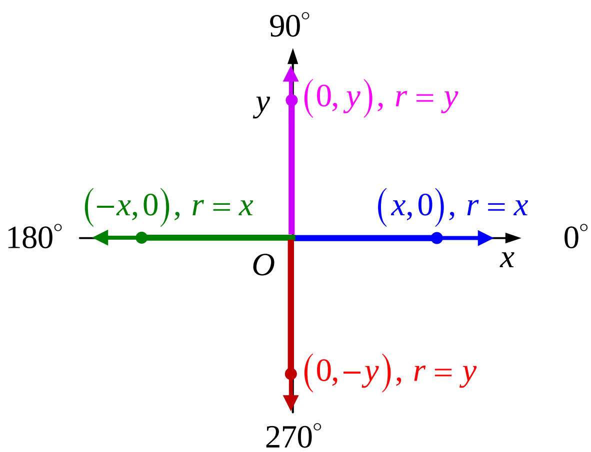
where $r = \sqrt{x^2 + y^2} > 0$

Example 1: Find the exact value of each of the six trigonometric functions of an angle in standard position whose terminal side contains the point $(4, 1)$.

Example 2: If the terminal side of angle θ lies on the line $4y + 3x = 0$, $x > 0$ find $\cot\theta + \cos\theta$.

II. Trigonometric Functions of Quadrantal Angles:

The terminal side of θ coincides with the positive x –axis. Let $P(x, y)$, $x > 0$, be any point on the x-axis. Then $y = 0$, and $r = x$: The value of the six trigonometric functions of 0° are:



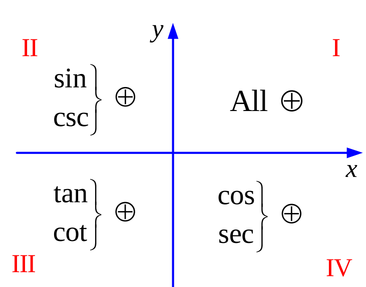
1. $\sin 0^\circ = \frac{0}{r} = 0$
2. $\cos 0^\circ = \frac{x}{r} = \frac{x}{x} = 1$
3. $\tan 0^\circ = \frac{0}{x} = 0$
4. $\csc 0^\circ = \frac{r}{0}$ is undefined
5. $\sec 0^\circ = \frac{r}{x} = \frac{x}{x} = 1$
6. $\cot 0^\circ = \frac{x}{0}$ is undefined

θ	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
0°	0	1	0	undefined	1	undefined
90°	1	0	undefined	1	undefined	0
180°	0	-1	0	undefined	-1	undefined
270°	-1	0	undefined	-1	undefined	0

Table 1: Values of Trigonometric Functions for Quadrantal Angles

III. Sign of Trigonometric Functions:

The sign of a trigonometric function depends on the quadrant in which the terminal side of the angle lies.



sign of	I	II	III	IV
sin $\boldsymbol{\theta}$ and **csc** $\boldsymbol{\theta}$	positive	positive	negative	negative
cos $\boldsymbol{\theta}$ and **sec** $\boldsymbol{\theta}$	positive	negative	negative	positive
tan $\boldsymbol{\theta}$ and **cot** $\boldsymbol{\theta}$	positive	negative	positive	negative

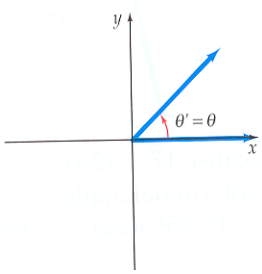
Table 2: Signs of the Trigonometric Functions

Example 3: Given $\sec\theta = 3$ and $\sin\theta < 0$ find $\tan\theta$ and $\csc\theta$.

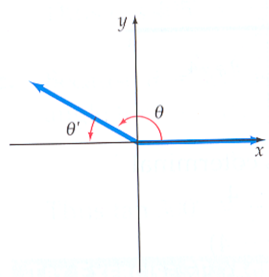
Example 4: If $\tan\beta = \dfrac{\sqrt{7}}{3}$ and $\sec\beta = -\dfrac{4}{3}$ find $\sin\beta$

IV. The Reference Angle:

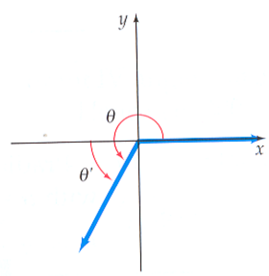
Given $\angle\theta$ in standard position, its **reference angle** θ' is the smallest positive angle formed by the terminal side of $\angle\theta$ and the x-axis.



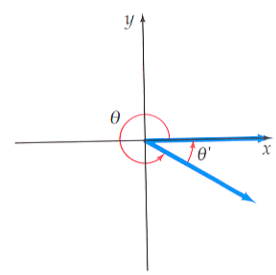
$\theta' = \theta$



$\theta' = 180^\circ - \theta$
$= \pi - \theta$



$\theta' = \theta - 180^\circ$
$= \theta - \pi$



$\theta' = 360^\circ - \theta$
$= 2\pi - \theta$

<u>*Note:*</u> If θ is negative or $> 360^\circ$, first find the coterminal angle and then find the reference angle.

Example 5: Find the reference angle of each of the following:

(a) $\theta = 135^\circ$ (b) $\theta = -210^\circ$ (c) $\theta = \dfrac{10\pi}{3}$ (d) $\theta = 924^\circ$ (e) $\theta = -6$ (f) $\theta = 30$

V. Reference Angle Theorem:

This theorem is used to find the trigonometric function of angles that are not acute. To evaluate such trigonometric function, do the following:

1. locate the quadrant of angle θ.
2. determine the sign $(+ \text{ or } -)$ of the trig function.
3. find the reference angle θ'
4. then, write as $\text{trig}\,\theta = \text{trig}\,\theta'$ or $-\text{trig}\,\theta'$ using correct sign.

Example 7: Evaluate each function

(a) $\sin 330^\circ$ (b) $\cos 405^\circ$ (c) $\tan 240^\circ$ (d) $\csc\frac{4\pi}{3}$ (e) $\cot\left(-\frac{\pi}{3}\right)$ (f) $\sec 765^\circ$

Example 8: Find the exact value of each of the following expression

(a) $\sin 210^\circ - \cos 330^\circ \tan 330^\circ$

(b) $\sin^2\left(\frac{5\pi}{4}\right) + \cos^2\left(\frac{5\pi}{4}\right)$

(c) $\tan^2\left(\frac{7\pi}{4}\right) - \sec^2\left(\frac{7\pi}{4}\right)$