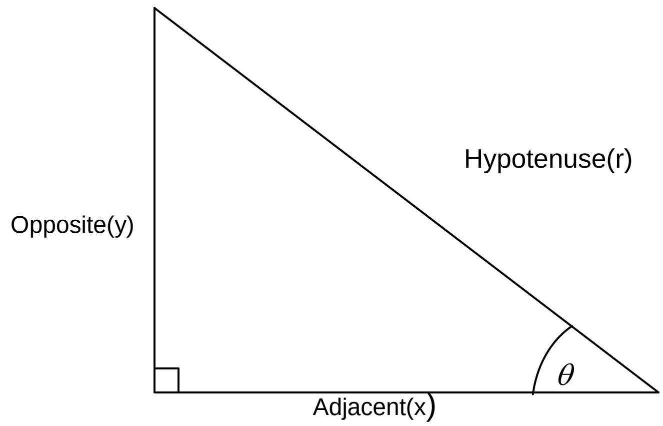
5.2 RIGHT TRIANGLE TRIGONOMETRY

Trigonometry is study involving triangle measurement. Right triangles are used to study functions that involve trigonometry (trigonometric functions)

Trigonometric function is a ratio of any two sides of a right angle triangle.



$$x^2 + y^2 = r^2 \qquad \text{(Pythagoras theorem)}$$

The 6 ratios defined for a right triangle are:

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, $\tan\theta = \frac{y}{x}$, $\csc\theta = \frac{r}{y}$,

$\sec\theta = \frac{r}{x}$ $\cot\theta = \frac{x}{y}$

Example 1

Let θ be an acute angle such that $\sin\theta = \frac{\sqrt{5}}{3}$ find:

a) $\sec\theta + \tan\theta$ b) $(\cot\theta)^2 + (\csc\theta)^2$

Solution

a) $\sin\theta = \frac{y}{r} \Rightarrow 3^2 = \left(\sqrt{5}\right)^2 + x^2 \Rightarrow x = 2$

$\Rightarrow \sec\theta = \frac{r}{x} = \frac{3}{2}$ and $\Rightarrow \tan\theta = \frac{y}{x} = \frac{\sqrt{5}}{2}$

$\Rightarrow \sec\theta + \tan\theta = \frac{3+\sqrt{5}}{2}$

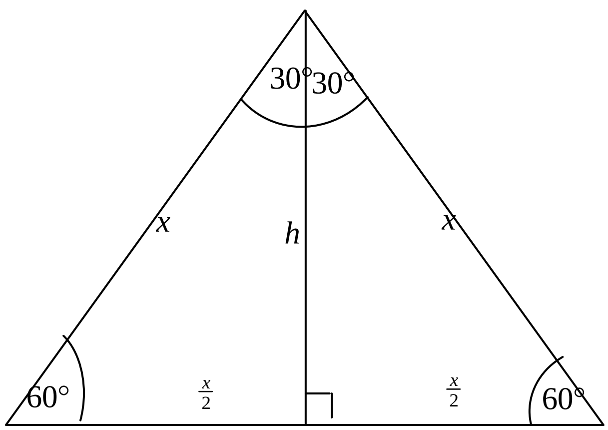
b) $\cot\theta = \frac{x}{y} = \frac{2}{\sqrt{5}} \Rightarrow (\cot\theta)^2 = \frac{4}{5}$

$$\csc\theta = \frac{r}{y} = \frac{3}{\sqrt{5}} \Rightarrow (\csc\theta)^2 = \frac{9}{5}$$

$$\Rightarrow (\cot\theta)^2 + (\csc\theta)^2 = \frac{4}{5} + \frac{9}{5} = \frac{13}{5}$$

Trigonometric ratios of special acute angles

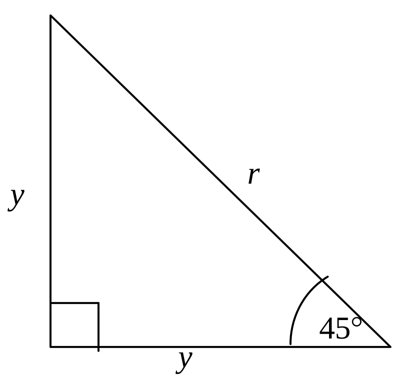
The equilateral and isosceles triangles shown below illustrates how trigonometric ratios for angles $30°$ $45°$ and $60°$ are derived.



Note: $h = \sqrt{x^2 - (\frac{x}{2})^2} = \frac{\sqrt{3}x}{2}$

$$\sin 30^\circ = \frac{\frac{x}{2}}{x} = \frac{1}{2}, \ \sin 60^\circ = \frac{h}{x} = \frac{\frac{\sqrt{3}x}{2}}{x} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\frac{x}{2}}{x} = \frac{1}{2} \quad , \qquad \cos 30^\circ = \frac{h}{x} = \frac{\sqrt{3}}{2}$$



$$\sin 45° = \cos 45° = \frac{y}{r} = \frac{y}{\sqrt{y^2 + y^2}} = \frac{y}{y\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Using the above two triangles, the following are obtained;

1. $\sin 0^{\circ} = \cos 90^{\circ} = 0$
2. $\sin 30^{\circ} = \cos 60^{\circ} = \frac{1}{2}$
3. $\sin 45^{\circ} = \cos 45^{\circ} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$
4. $\sin 60^{\circ} = \cos 30^{\circ} = \frac{\sqrt{3}}{2}$
5. $\sin 90^{\circ} = \cos 0^{\circ} = 1$
6. $\tan 0^{\circ} = \cot 90^{\circ} = 0$
7. $\tan 30^{\circ} = \cot 60^{\circ} = \frac{\sqrt{3}}{3}$
8. $\tan 45^{\circ} = \cot 45^{\circ} = 1$
9. $\tan 60^{\circ} = \cot 30^{\circ} = \sqrt{3}$

Example 2

Find the exact values of;

a) $3\sin 45^{\circ} \cos 30^{\circ} - \cot 30^{\circ}$

b) $2\tan\frac{\pi}{4} + \cos\frac{\pi}{3}\csc\frac{\pi}{6}$

c) $\sqrt{2}\cos\frac{\pi}{3}\cot\frac{\pi}{4} + 2\sqrt{3}\sec\frac{\pi}{6}$

Solution

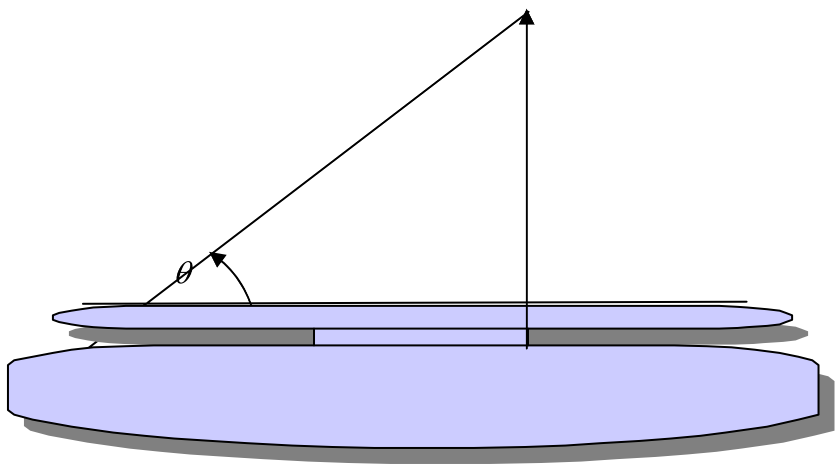
a) $3\sin 45^{\circ}\cos 30^{\circ} - \cot 30^{\circ} = 3.\frac{\sqrt{2}}{2}.\frac{\sqrt{3}}{2} - \sqrt{3} = \frac{3\sqrt{6} - 4\sqrt{3}}{4}$

b) $2\tan\frac{\pi}{4} + \cos\frac{\pi}{3}\csc\frac{\pi}{6} = 2.1 + \frac{1}{2}.2 = 3$

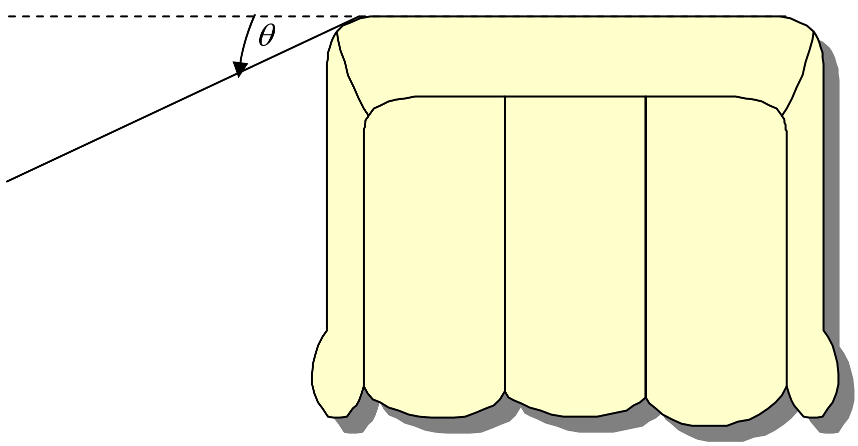
c) $\sqrt{2}\cos\frac{\pi}{3}\cot\frac{\pi}{4} + 2\sqrt{3}\sec\frac{\pi}{6} = \sqrt{2}.\frac{1}{2}.1 + 2\sqrt{3}.\frac{2\sqrt{3}}{3} = \frac{\sqrt{2} + 8}{4}$

Application of Trigonometric Functions of Acute Angles

Angle of Elevation is an angle measured upwards from the horizontal as shown in the figure below



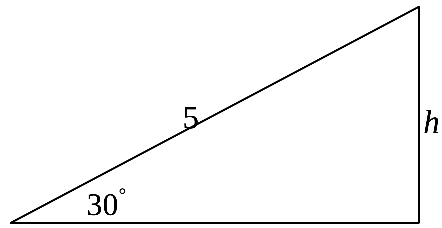
Angle of depression is an angle measured downwards from the horizontal as shown below



Example 3

A 5 meters ladder is resting against a wall and makes an angle of $30°$ with the ground. Find the height to which the ladder will reach the wall.

Solution

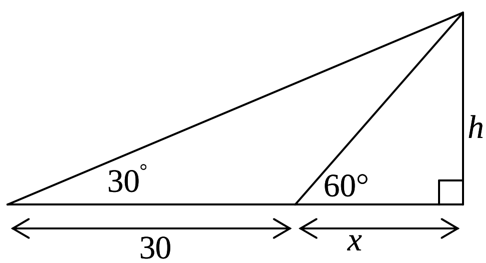


$$\frac{h}{5} = \sin 30^\circ = \frac{1}{2} \Rightarrow h = 2.5m$$

Example 4

Find the height of a building if the angle of elevation from the top of the building changes from 60° to 30° as the observer moves 30m further from the building.

Solution



$$\frac{h}{x} = \tan 60^\circ = \sqrt{3} \Rightarrow h = x\sqrt{3}$$

$$\frac{h}{x+30} = \tan 30^\circ = \frac{\sqrt{3}}{3} \Rightarrow h = \frac{x\sqrt{3}}{3} + \frac{30\sqrt{3}}{3}$$

$$\Rightarrow x\sqrt{3} = \frac{x\sqrt{3}}{3} + \frac{30\sqrt{3}}{3}$$

$$\Rightarrow x\sqrt{3} - \frac{x\sqrt{3}}{3} = \frac{30\sqrt{3}}{3} \Rightarrow x\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{30\sqrt{3}}{3}$$

$$\Rightarrow x\left(\sqrt{3}-\frac{\sqrt{3}}{3}\right)=\frac{30\sqrt{3}}{3}$$

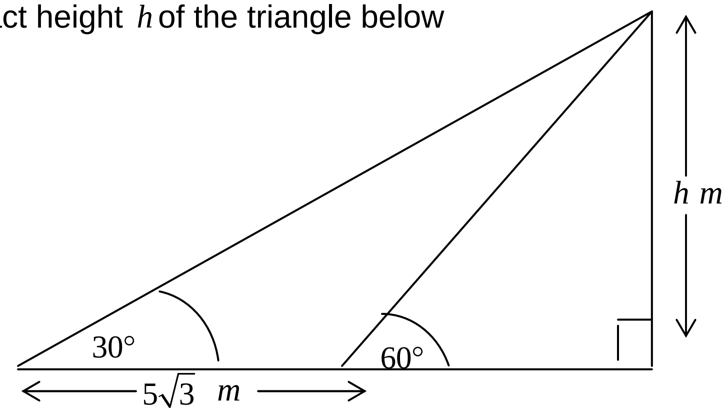
$$\Rightarrow x.\frac{2\sqrt{3}}{3}=\frac{30\sqrt{3}}{3}$$

$$\Rightarrow x=15\sqrt{3}$$

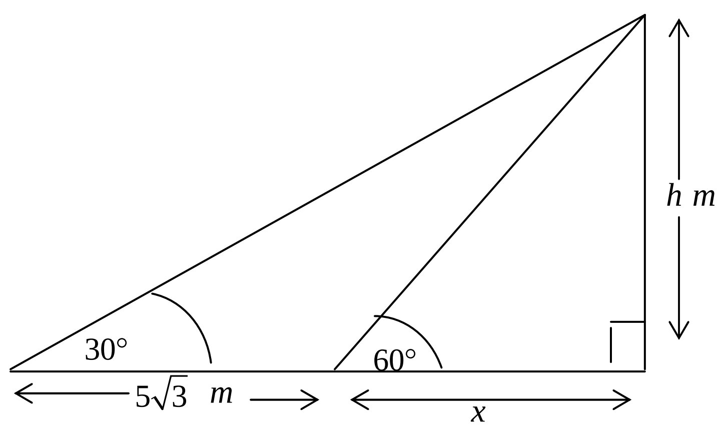
$$\Rightarrow h=15\sqrt{3}.\sqrt{3}=45$$

Example 5

Find the exact height h of the triangle below



Solution



$$\frac{h}{x}=\tan 60^{\circ}=\sqrt{3}\Rightarrow h=x\sqrt{3}$$

$$\frac{h}{x+5\sqrt{3}}=\tan 30^{\circ}=\frac{\sqrt{3}}{3}\Rightarrow h=\frac{x\sqrt{3}}{3}+5$$

$$\Rightarrow x\sqrt{3} = \frac{x\sqrt{3}}{3} + 5$$

$$\Rightarrow x\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = 5$$

$$\Rightarrow x.\frac{2\sqrt{3}}{3} = 5$$

$$\Rightarrow x = \frac{5\sqrt{3}}{2}$$

$$\Rightarrow h = \frac{5\sqrt{3}}{2}.\sqrt{3} = \frac{15}{2} = 7.5m$$