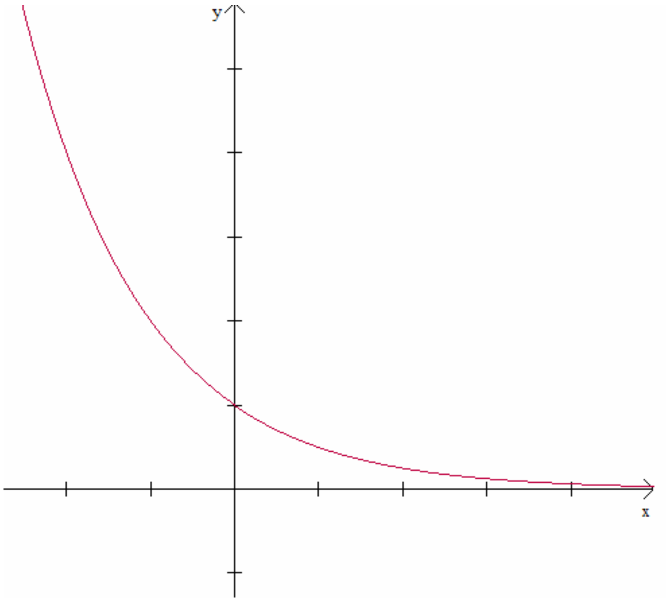
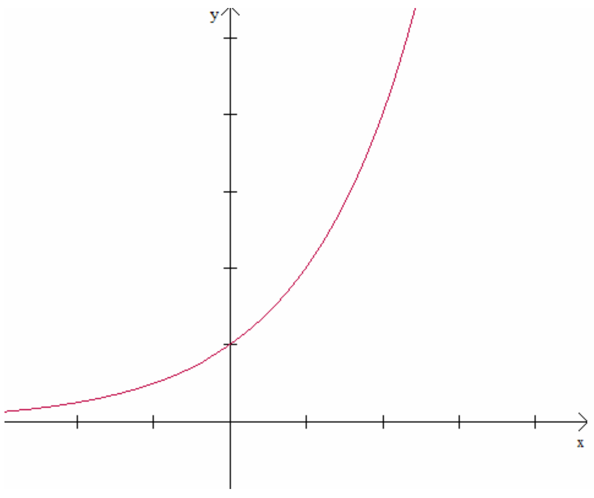
4.2 EXPONENTIAL FUNCTIONS AND THEIR APPLICATIONS

Functions of the type $f(x) = b^x$, x is real, $b > 0$ and $b \neq 1$ is called an exponential functions with the base b.
The graphs of exponential equations are of two types;

a) $0 < b < 1$



b) $b > 1$



Properties of graphs of Exponential functions:

1. The y intercept is always at (0,1).
2. Domain is $(-\infty, +\infty)$, Range is $(0, +\infty)$
3. Behaviour of the graph when
   a) $b > 1$

      $x-$ axis is an asymptote

as $x \to +\infty$, $y \to +\infty$

as $x \to -\infty$, $y \to 0$
An increasing function.

b) $0 < b < 1$
x – axis is an asymptote
as $x \to +\infty$, $y \to 0$

as $x \to -\infty$, $y \to +\infty$
A decreasing function

4. One-to one function.
5. No symmetry with respect to x, y or origin

Example 1

a) Evaluate $f(x) = \left(\frac{1}{2}\right)^x - 8$ at $x = -2$

b) Find domain and range of $f(x) = 4^{x-2} + 3$

Solution

a) $f(x) = \left(\frac{1}{2}\right)^{-2} - 8 = 4 - 8 = -4$

b) as $x \to +\infty$, $y \to 4^{\infty-2} + 3 = +\infty$

as $x \to -\infty$, $y \to 4^{-\infty-2} + 3 = 3$

$\Rightarrow$ Domain $(-\infty, +\infty)$ and Range $(3, +\infty)$

Example 2

a) Sketch the graph of i) $f(x) = \left(\frac{1}{2}\right)^x$ ii) $f(x) = (2)^x$

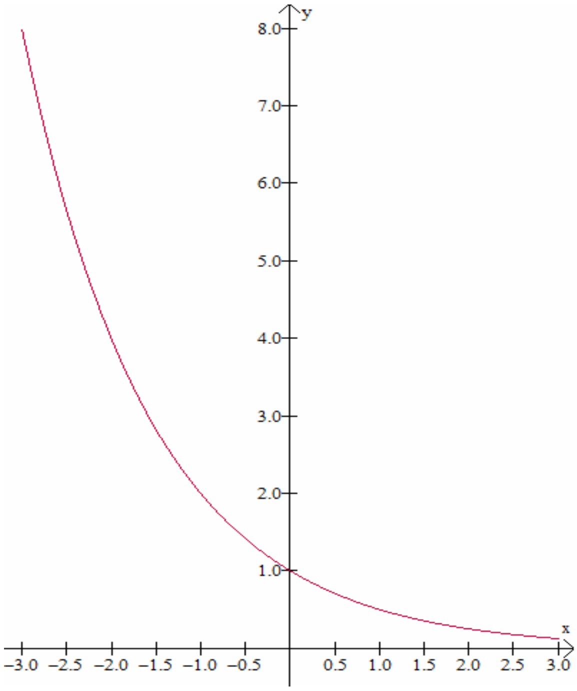
b) Explain how you would use the graph of $f(x) = \left(\frac{1}{2}\right)^x$ to draw the graph of

$f(x) = \left(\frac{1}{2}\right)^{1-x}$

Solution

a) Method 1

x	-3	-2	-1	0	1	2	3
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

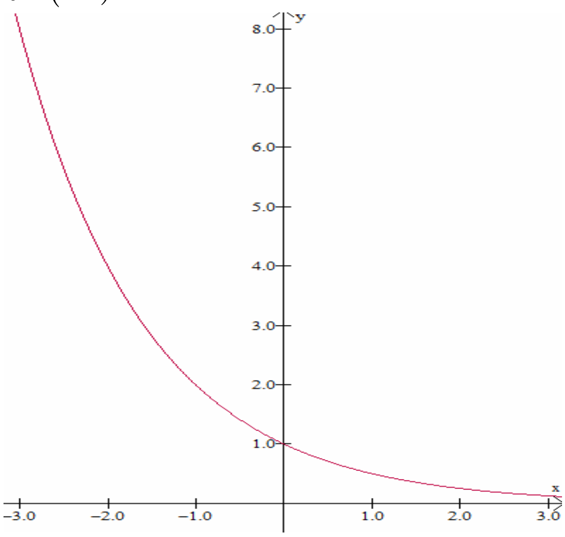


Method 2

as $x \to +\infty$, $y \to \left(\frac{1}{2}\right)^{\infty} = 0$

as $x \to -\infty$, $y \to \left(\frac{1}{2}\right)^{-\infty} = \infty$

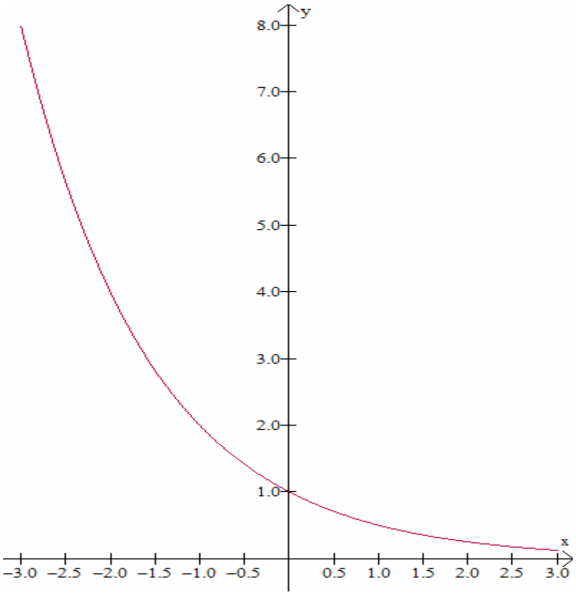
$y-$Intercept $(0,1)$



Note:

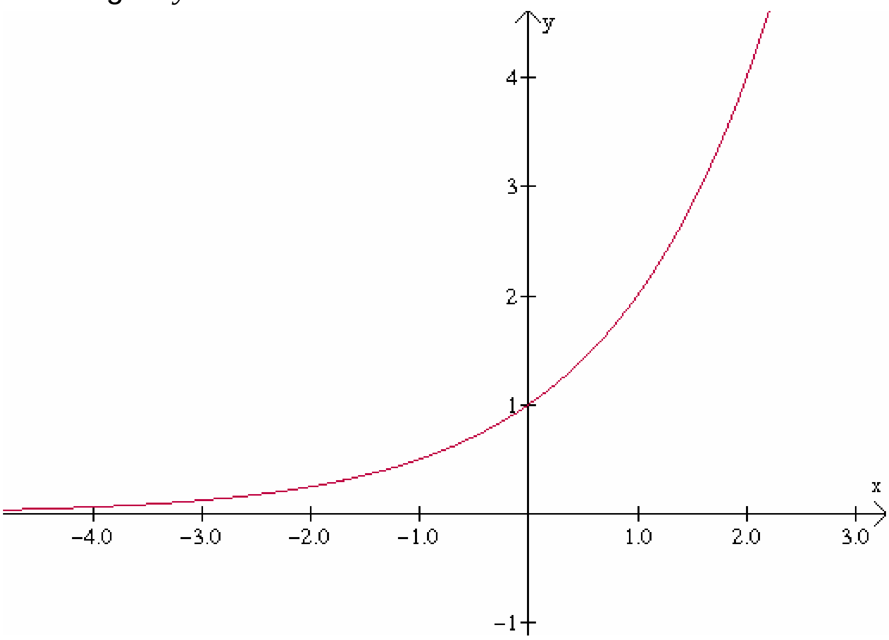
Although using tables is the easiest method for sketching graphs an alternative method of using the graph behaviour is often used whenever the domain values are irrational or very small decimal numbers.

b) The graph of $f(x) = \left(\frac{1}{2}\right)^{x}$ is shown below;



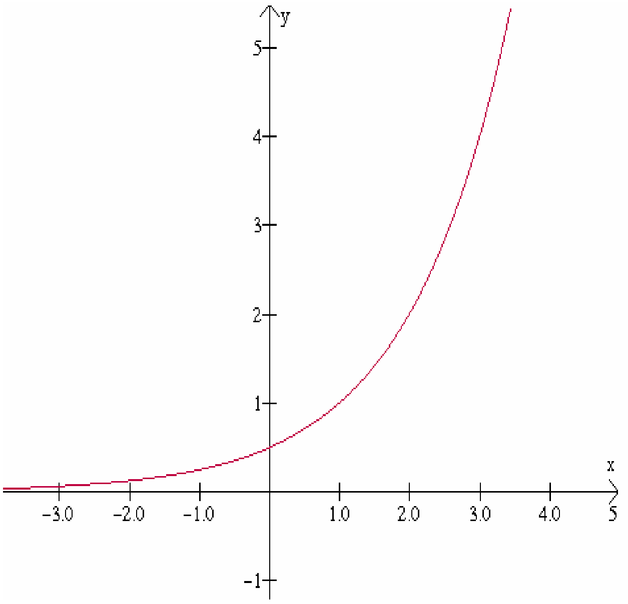
$f(x) = \left(\frac{1}{2}\right)^x$

The graph of $f(-x) = \left(\frac{1}{2}\right)^{-x}$ is obtained from the graph of $f(x) = \left(\frac{1}{2}\right)^x$ by reflecting in $y-$axis



$f(-x) = \left(\frac{1}{2}\right)^{-x}$

The graph of $f(1-x) = f(-x+1) = \left(\frac{1}{2}\right)^{1-x}$ is obtained from the graph of $f(-x) = \left(\frac{1}{2}\right)^{-x}$ by horizontal shift of 1 unit to the right as shown below.



$$f(-x+1)=\left(\tfrac{1}{2}\right)^{1-x}$$

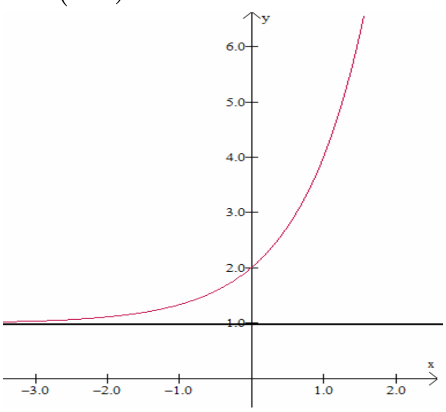
Example 3:
Sketch the graphs whose equations are given below. In each case state the asymptotes, domain and range.

a) $f(x)=3^x+1$ b) $f(x)=2^{(x-1)}$ c) $f(x)=\left(\tfrac{2}{5}\right)^{-x}$ d) $f(x)=-7^x$ e) $f(x)=\left(\dfrac{1}{2}\right)^{-x-1}+1$

Solution

a) $f(x)=3^x+1$

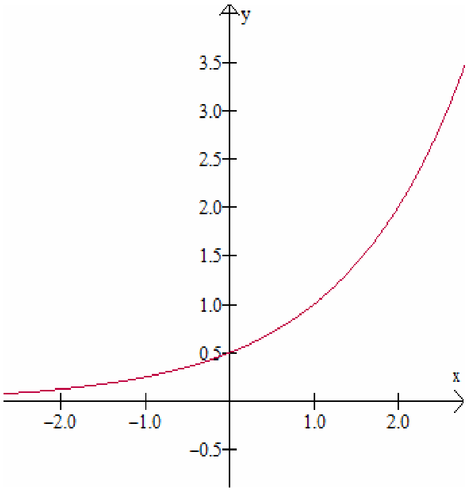
as $x\to+\infty$, $y\to+\infty$
as $x\to-\infty$, $y\to1$
$y-$Intercept $(0,2)$



Domain is $(-\infty,+\infty)$, Range is $(1,+\infty)$ and the equation of asymptote is $y=1$

b) $f(x)=2^{(x-1)}$

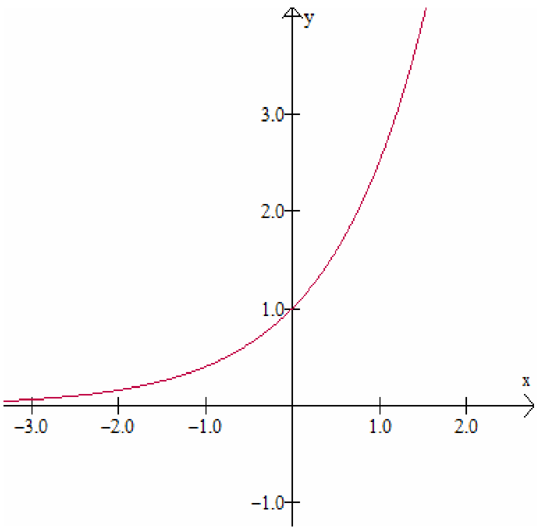
as $x \to +\infty$, $y \to +\infty$
as $x \to -\infty$, $y \to 0$
$y-$Intercept $\left(0, \frac{1}{2}\right)$



Domain is $(-\infty,+\infty)$, Range is $(0,+\infty)$ and the equation of asymptote is $y=0$

c) $f(x)=\left(\frac{2}{5}\right)^{-x}$

as $x \to +\infty$, $y \to +\infty$
as $x \to -\infty$, $y \to 0$
$y-$Intercept $(0,1)$

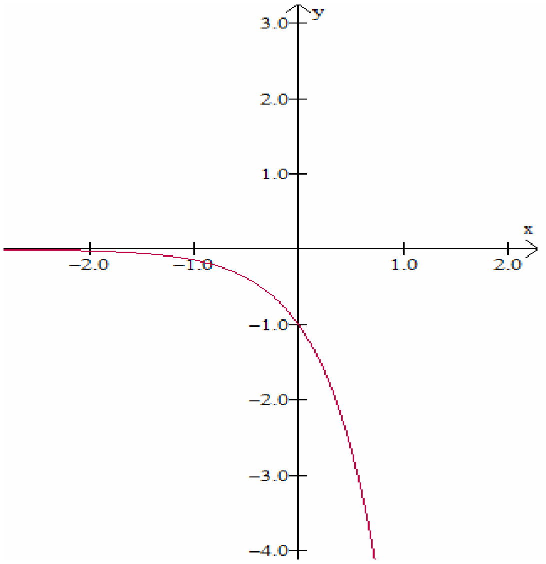


Domain is $(-\infty,+\infty)$, Range is $(0,+\infty)$ and equation of the asymptote is $y=0$

d) $f(x)=-7^{x}$

as $x \to +\infty$, $y \to -\infty$
as $x \to -\infty$, $y \to 0$

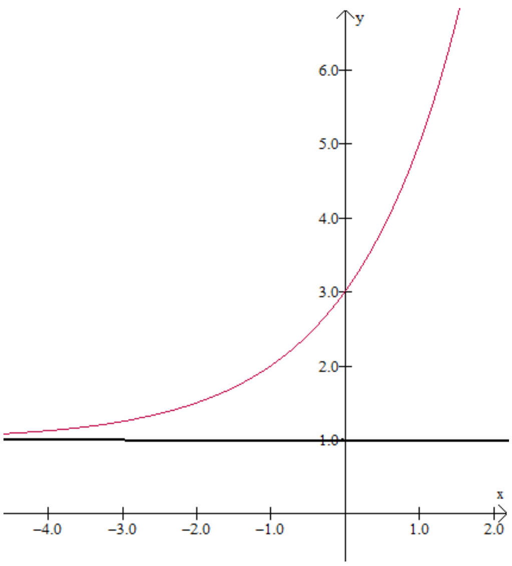
$y-$Intercept $(0,-1)$



Domain is $(-\infty,+\infty)$, Range is $(-\infty,0)$ and equation of the asymptote is $y=0$

e) $f(x)=\left(\frac{1}{2}\right)^{-x-1}+1$

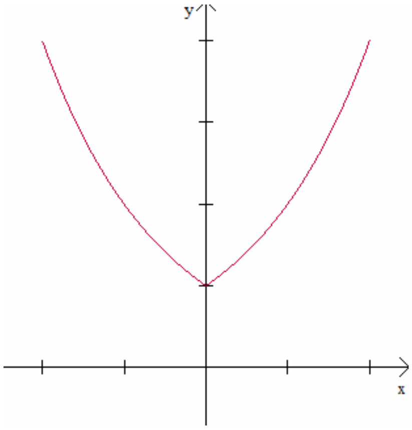
as $x\to+\infty$, $y\to+\infty$
as $x\to-\infty$, $y\to1$
$y-$Intercept $(0,3)$



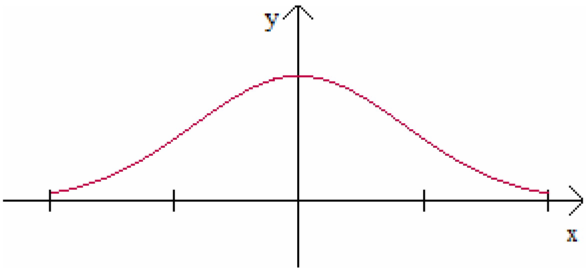
Domain is $(-\infty,+\infty)$, Range is $(1,+\infty)$ and equation of the asymptote is $y=1$

Effect of Absolute Value and Squaring

1. The graphs of $f(x)=b^{|x|}$ or $f(x)=b^{x^2}$ is symmetric with respect to $y-$axis and has intercept at (0,1) .
   a) If $b>1$ the graph of $f(x)$ has the shape shown below;

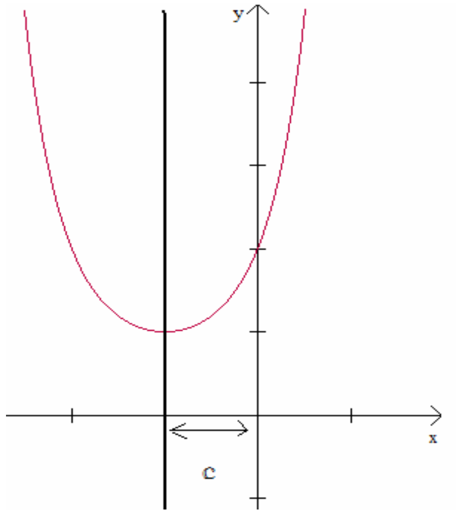


   b) If $0<b<1$ the graph of $f(x)$ has the shape shown below

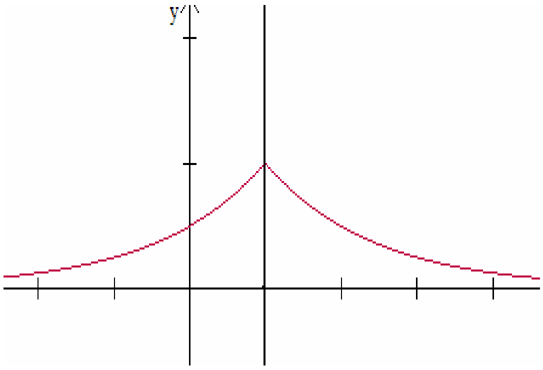


2. The graph of $f(x)=b^{|x+c|}$ has intercept at $\left(0,b^{|c|}\right)$ and $f(x)=b^{(x+c)^2}$ has intercept at $\left(0,b^{c^2}\right)$. Both have graphs are symmetric with respect to the line $x=-c$ axis and

   a) If $b>1$ the graph of $f(x)$ has the shape shown below;

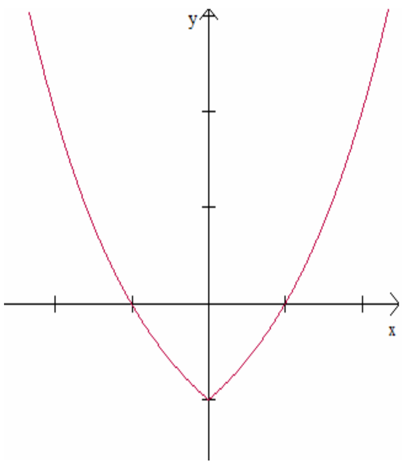


b) If $0 < b < 1$ the graph of $f(x)$ has the shape shown below

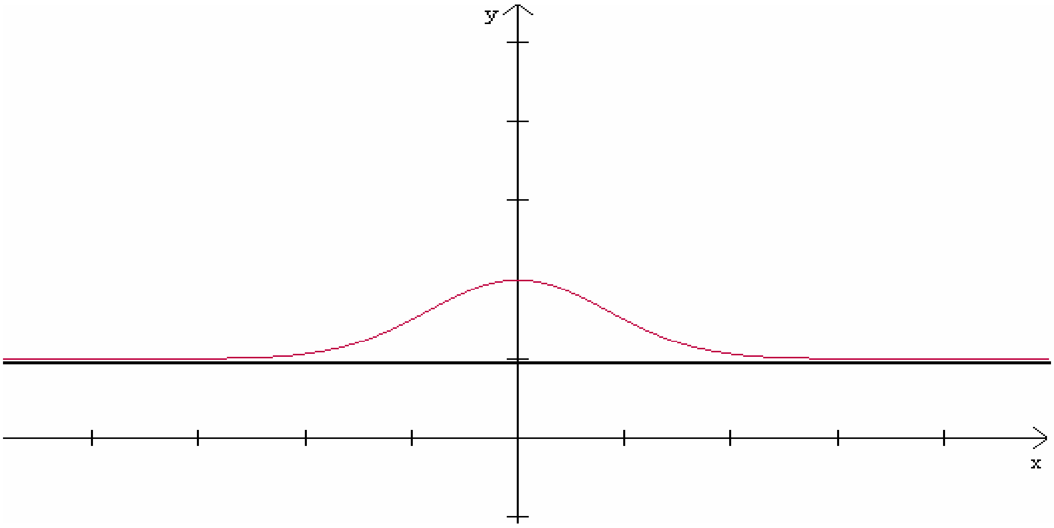


3. The graphs of $f(x) = b^{|x|} + c$ or $f(x) = b^{x^2} + c$ is symmetric with respect to the $y-$ axis and has intercept at $(0, 1+c)$

a) If $b > 1$ the graph of $f(x)$ has the shape shown below;



b) If $0 < b < 1$ the graph of $f(x)$ has the shape shown below



Example4:

Sketch the graphs of;

a) $f(x)=3^{|x|}$ b) $f(x)=10^{-|x|}$ c) $f(x)=\left(\frac{1}{4}\right)^{|x|}+3$

d) $f(x)=5^{x^2}$ e) $f(x)=\left(\frac{1}{2}\right)^{-x^2}+2$ f) $f(x)=\left(\frac{1}{3}\right)^{|x+2|}$

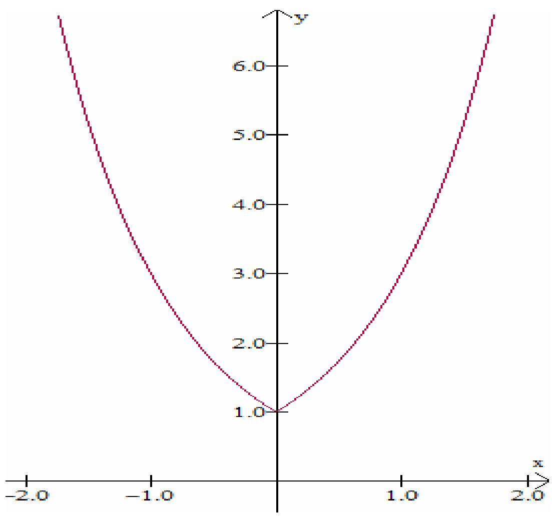
Solution

a) $f(x)=3^{|x|}$

as $x \to +\infty$, $y \to +\infty$

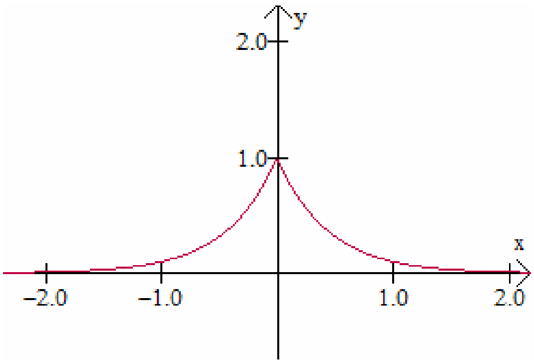
as $x \to -\infty$, $y \to +\infty$

$y-$Intercept $(0,1)$



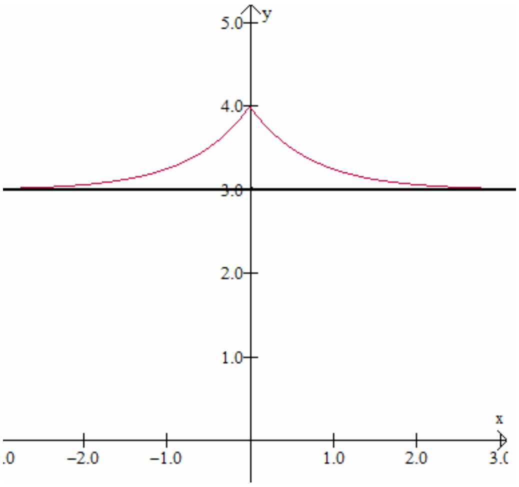
b) $f(x)=10^{-|x|}$

as $x \to +\infty$, $y \to 0$
as $x \to -\infty$, $y \to 0$
y – Intercept $(0,1)$



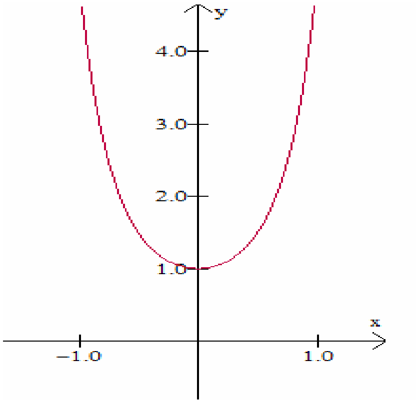
c) $f(x) = \left(\frac{1}{4}\right)^{|x|} + 3$

as $x \to +\infty$, $y \to 3$
as $x \to -\infty$, $y \to 3$
y – Intercept $(0,4)$



d) $f(x) = 5^{x^2}$

as $x \to +\infty$, $y \to +\infty$
as $x \to -\infty$, $y \to +\infty$
y – Intercept $(0,1)$

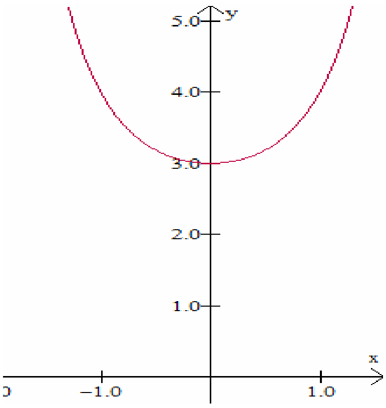


e) $f(x)=\left(\frac{1}{2}\right)^{-x^2}+2$

as $x\to+\infty$, $y\to+\infty$

as $x\to-\infty$, $y\to+\infty$

$y-$Intercept $(0,3)$



f) $f(x)=\left(\frac{1}{3}\right)^{|x+2|}$

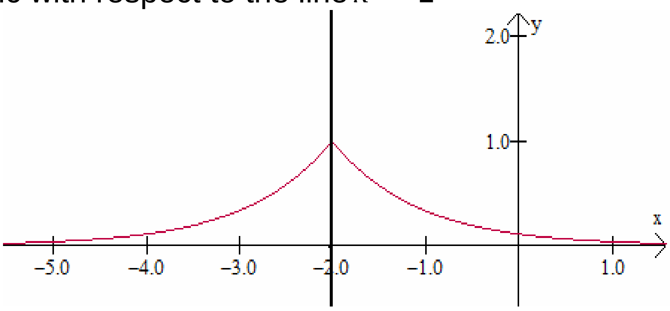
Method 1

as $x\to+\infty$, $y\to 0$

as $x\to-\infty$, $y\to 0$

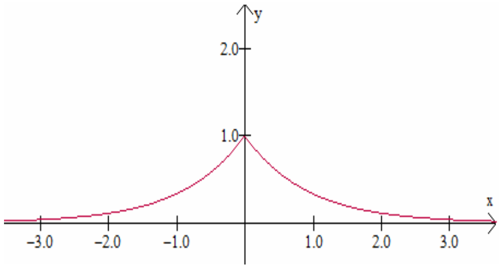
$y-$Intercept $\left(0,\frac{1}{9}\right)$

Symmetric with respect to the line $x=-2$

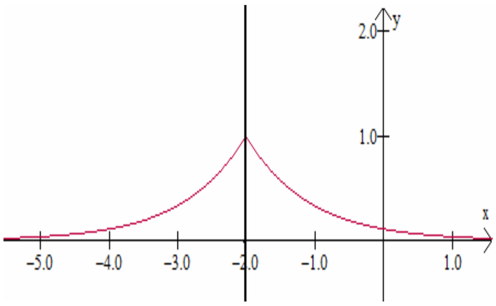


Method 2

The graph of $f(x)=\left(\frac{1}{3}\right)^{|x+2|}$ is the same as the graph $f(x)=\left(\frac{1}{3}\right)^{|x|}$ shifted 2 units horizontally to the left.



$$f(x)=\left(\frac{1}{3}\right)^{|x|}$$



$$f(x)=\left(\frac{1}{3}\right)^{|x+2|}$$

Note: *when a graph is shifted horizontally* a *units the vertical axis of symmetry also moves* a *in the same direction.*

Natural Exponential functions

An exponential function with the base $e \simeq 2.72$ is called a natural exponential function.

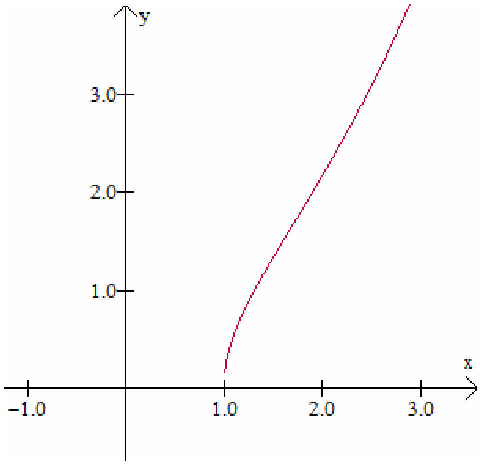
Example 5:
Sketch the graphs of;

a) $f(x)=\sqrt{e^x - e}$ b) $f(x)=\frac{e^x+e^{-x}}{2}$ c) $f(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}$

Solution

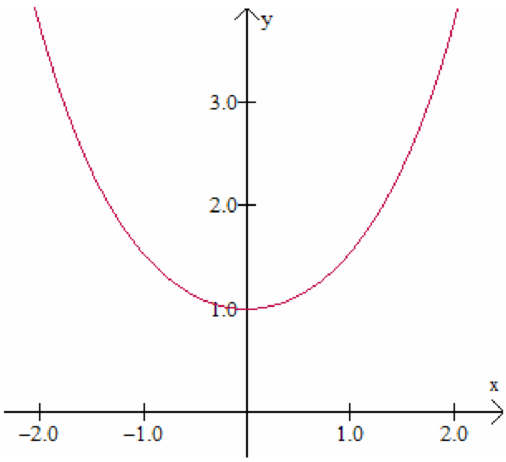
a) $f(x)=\sqrt{e^x - e}$

as $x \to +\infty$, $y \to +\infty$
as $x \to -\infty$, y is undefined.
$x-$Intercept $(1,0)$
Domain $\Rightarrow e^x - e \geq 0 \Rightarrow x \geq 1$



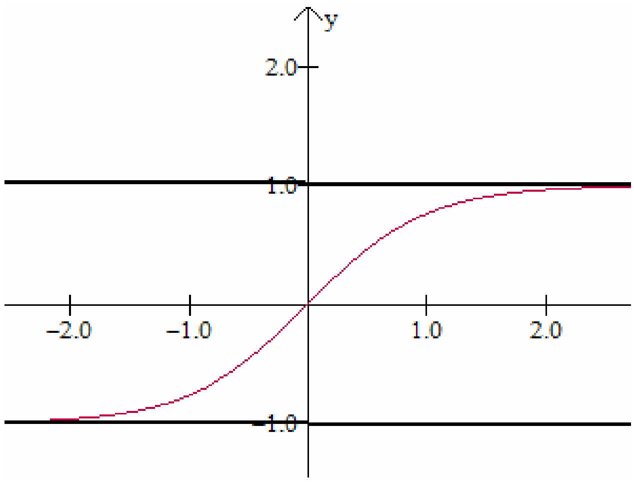
b) $f(x) = \dfrac{e^x + e^{-x}}{2}$

as $x \to +\infty$, $y \to +\infty$
as $x \to -\infty$, $y \to +\infty$
$y-$Intercept $(0,1)$



c) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

as $x \to +\infty$, $y \to +1$
as $x \to -\infty$, $y \to -1$
$y-$Intercept $(0,0)$
Domain is $(-\infty, +\infty)$
Symmetric with respect to the origin



Example 6

Find the domain and the range of;

a) $f(x)=\sqrt{1-e^{2x}}$ b) $f(x)=\dfrac{3x}{e^x-1}$

Solution

a) $\Rightarrow 1-e^{2x}\geq 0 \Rightarrow x\leq 0$

Domain is $(-\infty,0]$ and Range is $[0,+\infty)$

b) $\Rightarrow 3x$ is real and $e^x-1\neq 0$

$\Rightarrow x$ is real and $x\neq 0$

Domain is $(-\infty,0)\cup(0,+\infty)$ and Range is $(-\infty,0)\cup(0,+\infty)$

Example 7

Determine the horizontal asymptotes of $f(x)=\dfrac{e^x+e^{-x}}{e^x-e^{-x}}$

Solution

as $x\to+\infty$, $y\to+1$

as $x\to-\infty$, $y\to-1$

The horizontal asymptotes are $y\to\pm 1$

Exponential Inequalities

Theorem:

If $b^x\leq b^y$ then (i) $x\geq y$ when $0<b<1$

(ii) $x\leq y$ when $b>1$

Example 8:

Solve the exponential inequalities;

a) $3^{(x-1)} \le 9^{(x-3)}$ b) $\left(\dfrac{1}{2}\right)^{2x} \ge \left(\dfrac{1}{2}\right)^{(3x-1)}$ c) $(4.5)^{2x-1} \ge (4.5)^{4-3x}$

d) $e^{4x-3} \le e^{x}$ e) $\left(\frac{1}{4}\right)^{x-3} \le \left(\frac{1}{2}\right)^{(x+2)}$

Solution

a) $3^{(x-1)} \le 3^{2(x-3)}$

$3^{(x-1)} \le 3^{(2x-6)} \Rightarrow x-1 \le 2x-6$

$\Rightarrow x \ge 5$

b) $\Rightarrow 2x \le 3x-1$

$\Rightarrow x \ge 1$

c) $\Rightarrow 2x-1 \ge 4-3x$

$\Rightarrow x \ge 1$

d) $\Rightarrow 4x-3 \le x$

$\Rightarrow x \le 1$

e) $\Rightarrow 2x-6 \ge x+2$

$\Rightarrow x \ge 8$