— 3.3. Zeros of Polynomial Functions —

Objectives.

- Give some techniques to solve some higher degree polynomials, equations.
- Use zeros to factor a polynomial.
- How to find the multiplicity of a zero.

Factor Theorem

$(x-c)$ is a factor of $p(x) \iff p(c)=0$

Exp1. Determine if the second polynomial is a factor of the first polynomial.

a) $p(x) = x^6 - x^4 + 2x^2 - 2$, $d(x) = x - 1$

b) $p(x) = 5x^4 + 16x^3 - 15x^2 + 8x + 16$, $d(x) = x + 4$

Sol^n. a) $p(1) = 1^6 - 1 + 2(1) - 2 = 0$

$\rightarrow x - 1$ is a factor of $p(x)$

or

$$\begin{array}{r|rrrrrrr} 1 & 1 & 0 & -1 & 0 & 2 & 0 & -2 \\ & & 1 & 1 & 0 & 0 & 2 & 2 \\ \hline & 1 & 1 & 0 & 0 & 2 & 2 & \boxed{0} \end{array}$$

b)

$$\begin{array}{r|rrrrr} -4 & 5 & 16 & -15 & 8 & 16 \\ & & -20 & 16 & -4 & -16 \\ \hline & 5 & -4 & 1 & 4 & \boxed{0} \end{array}$$

Exp 2 Factor $p(x) = 6x^3 + 19x^2 + 2x - 3$ into linear factors if -3 is a zero of f.

Let us divide $p(x)$ by $x+3$,

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 2 & -3 \\ & & -18 & -3 & 3 \\ \hline & 6 & 1 & -1 & 0 \end{array}$$

$$\Rightarrow \quad p(x) = (x+3)(6x^2 + x - 1)$$
$$= (x+3)(3x-1)(2x+1)$$

Multiplicity of a zero:

A zero c of $p(x)$ has multiplicity k if $p(x) = (x-c)^k q(x)$ & $q(c) \neq 0$

Exp3. Find the zeros & their multiplicities of

$$p(x) = 2(x-2)(x+1)^2(x-4)^5$$

The zeros	the multiplicity
$c = 2$	$k = 1$
$c = -1$	$k = 2$
$c = 4$	$k = 5$

Exp 4. Find all zeros & their multiplicities (3.3 p3)

of $p(x) = (x^2 - x - 2)^5 (x - 1 + \sqrt{3})^2$

$$p(x) = \left[(x-2)(x+1)\right]^5 \left(x - (1-\sqrt{3})\right)^2$$

$$= (x-2)^5 (x+1)^5 \left(x - (1-\sqrt{3})\right)^2$$

Zero	mult.
2	$k=5$
-1	$k=5$
$1-\sqrt{3}$	$k=2$

Exp 5. Knowing that $c = -1$ is a zero of mult. 3

of $p(x) = 2x^4 + x^3 - 9x^2 - 13x - 5$

Factor $p(x)$ into linear factors.

$$\begin{array}{r|rrrrr}
-1 & 2 & 1 & -9 & -13 & -5 \\
 & & -2 & 1 & 8 & 5 \\
\hline
-1 & 2 & -1 & -8 & -5 & \boxed{0} \\
 & & -2 & 3 & 5 & \\
\hline
-1 & 2 & -3 & -5 & \boxed{0} & \\
 & & -2 & 5 & & \\
\hline
 & 2 & -5 & 0 & &
\end{array}$$

$\Rightarrow 2x - 5$

$$\Rightarrow \boxed{p(x) = (x+1)^3 (2x-5)}$$

Fundamental Theorem of Algebra.

Any nonzero polynomial of degree n with real or complex coefficients has exactly n zeros (real or complex) (every zero is counted the number of its multiplicity)

In other words, the number of the multiplicities of the zeros is n.

If we count each zero only one time, the number of zeros is less than or equal to n.

Exp. $p(x) = (x-1)^2(x+2)^3$

$n=5$, $\quad$ 1 has $k=2$, $\quad$ -2 has $k=3$

$$\boxed{2+3=5}$$

Descartes' Rule of Signs DRS

Let $p(x)$ be a poly. in standard form like $p(x) = 10x^6 - 9x^5 - 14x^4 - 8x^3 - 18x^2 + x + 6$

The nbr of sign variations in coeff of x is

$V(p(x)) = 2$

$p(-x) = 10(-x^6) - 9(-x^5) - 14(-x)^4 - 8(-x)^3 - 18(-x)^2 + (-x) + 6$

$+ \quad + \quad - \quad + \quad - \quad - \quad +$

$V(p(-x)) = 4$

Descartes' Rule of Sign Theorem.

If $p(x)$ is a poly with real coeff, & nonzero constant term a, then

1) The nbr of positive zeros $= V(p(x))$ - even nbr

2) The nbr of negative zeros $= V(p(-x))$ - even nbr

Exp: for $p(x) = 10x^6 - 9x^5 - 4x^4 - 8x^3 - 18x^2 + x + 6$

$p(-x) = + \quad + \quad - \quad + \quad - \quad - \quad +$

$V(p)(x) = 2, \quad V(p(-x)) = 4.$

So nbr of positive zeros is 2 or 0.

nbr of negative zeros is 4, 2 or 0

b) For the above poly find all possibilities of positive, negative & nonzero complex roots.

deg = 6 $\Rightarrow$ total nbr of zeros = 6

Total	positive	neg	nonreal complex
6	2	4	0
6	2	2	2
	2	0	4
	0	4	2
	0	2	4
	0	0	6

Rational Zeros Theorem.

Let $p(x) = a_n x^n + \cdots + a_1 x + a_0$ be a poly. with <u>integer</u> coefficients. Then

1) If p/q is a rational zero of $p(x)$ then $\underline{p}$ is a factor of $\underline{a_0}$ & $\underline{q}$ is a factor of a_n

In other words

$$\{\text{rational Zeros of } p(x)\} \subseteq \left\{ p/q : \begin{array}{l} p \text{ factor of } a_0 \\ q \quad '' \quad \text{of } a_n \end{array} \right\}$$

factor : means divisor (positive or negative).

<u>Exp</u> List all possible rational zeros of

$$p(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

Sol^n: $a_0 = 2 \Rightarrow p =$ factors of $a_0 = 2$

$= \pm 1, \pm 2$

$a_n = 6 \Rightarrow q$: factors of $a_n = 6$

$= \pm 1, \pm 2, \pm 3, \pm 6$

List of all possible zeros $= \frac{p}{q}$

$= \pm 1, \pm 2, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{1}{6}$.

Exp. Do the same for $p(x) = 4x^5 + 2x^2 - 3x + 3$

Boundedness Theorem.

If $p(x)$ is a poly. with real coeff, $a_n > 0$, then

1) If $c > 0$ & the division by $(x - c)$, we have

$$\begin{array}{c|cccc} c & a_n & a_{n-1} & & a_0 \\ & & & & \\ \hline & b_n & b_{n-1} & & b_0 \\ & \geq 0 & \geq 0 & & \geq 0 \end{array}$$

$\Rightarrow$ $p(x)$ has no real zero $> c$

(c upper bound)

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2) If $c < 0$ &

$$\begin{array}{c|ccccc} c & a_n & a_{n-1} & a_{n-2} & & a_0 \\ & & & & & \\ \hline & b_n & b_{n-1} & b_{n-2} & \cdots & b_0 \\ & \geq 0 & \leq 0 & \geq 0 & & \end{array}$$

alternating in sign.

$\Rightarrow$ $p(x)$ has no real zero $< c$

(c: lower bound)

Exp. Let $p(x) = 2x^3 + 7x^2 - 4x - 14$

Show that there is no real zero > 2

& no zero < -4

$$\begin{array}{r|rrrr} 2 & 2 & 7 & -4 & -14 \\ & & 4 & 22 & 36 \\ \hline & 2 & 11 & 18 & 22 \end{array}$$

all nonnegative

$\Rightarrow$ No zero > 2

$$\begin{array}{r|rrrr} -4 & 2 & 7 & -4 & -14 \\ & & -8 & 4 & 0 \\ \hline & 2 & -1 & 0 & -14 \\ & \geq 0 & \leq 0 & \geq 0 & \leq 0 \end{array}$$

alternating in sign $\Rightarrow$ No zero < -4.

Conjugate Pair Theorem.

If $p(x)$ is a polynomial with real coefficients, & $z = a + ib$ is a zero of $p(x)$ then the conjugate $\bar{z} = a - ib$ is also a zero.

Exp. Given that i is a zero of $p(x) = x^4 + 5x^2 + 4$

Find all zeros of $p(x)$.

Coeff real, i zero $\Rightarrow$ $(-i)$ is a zero also

$$\begin{array}{r|rrrrr}
i & 1 & 0 & 5 & 0 & 4 \\
 & & i & -1 & 4i & -4 \\
\hline
-i & 1 & i & 4 & 4i & \boxed{0} \\
 & & -i & 0 & -4i & \\
\hline
 & 1 & 0 & 4 & 0 &
\end{array}$$

$$\Rightarrow p(x) = (x-i)(x+i)(x^2+4)$$

$x = i$, $x = -i$ $\quad x^2 = -4 \Rightarrow x = \pm 2i$

Exp. Find all zeros of $p(x) = x^3 - 7x^2 + 17x - 15$ given $(2-i)$ is a zero of $p(x)$.

Solving higher degree polynomial Equations

Exp Find all rational zeros of

$$p(x) = 2x^3 + x^2 - 13x + 6$$

Sol^n $V(p(x)) = 2$

$$p(-x) = - \quad + \quad + \quad +$$

2 or 0 posi & 1 neg.

$a_0 = 6 \qquad p = \pm 1, \pm 2, \pm 3, \pm 6$

$a_n = 2 \qquad q = \pm 1, \pm 2$

$$p/q = \pm 1, \pm 2, \pm 3, \pm 6, \pm 1/2, \pm 3/2 .$$

We check $x = 1$

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -13 & 6 \\ & & 1 & 3 & -10 \\ \hline & 2 & 3 & -10 & -4 \end{array} \qquad \begin{array}{r|rrrr} 2 & 2 & 1 & -13 & 6 \\ & & 4 & 10 & -6 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$$\Rightarrow p(x) = (x-2)(2x^2 + 5x - 3)$$

$$= (x-2)(2x-1)(x+3)$$

$$\Rightarrow \boxed{x = 2, \quad x = 1/2, \quad x = -3}$$

Exp Find all zeros of $P(x) = x^4 - 5x^3 - 5x^2 + 23x + 10$

Sol^n. DRS

$V_{p(x)} = 2 \Rightarrow$ 2 or 0 positive zeros

$p(-x) = + + - - + \Rightarrow$ 2 or 0 neg zero

Rational Zeros.

$a_0 = 10 \Rightarrow p = \pm 1, \pm 2, \pm 5, \pm 10$

$a_n = 1 \Rightarrow q = \pm 1$

$\Rightarrow \frac{p}{q} = \pm 1, \pm 2, \pm 5, \pm 10$

Checking

$$\begin{array}{r|rrrrr} 1 & 1 & -5 & -5 & 23 & 10 \\ & & 1 & -4 & -9 & +14 \\ \hline & 1 & -4 & -9 & 14 & 24 \neq 0 \end{array}$$

$$\begin{array}{r|l} 2 & \\ \hline & 12 \end{array}$$

$$\begin{array}{r|rrrrr} 5 & 1 & -5 & -5 & 23 & 10 \\ & & 5 & 0 & -25 & -10 \\ \hline -2 & 1 & 0 & -5 & -2 & \boxed{0} \\ & & -2 & 4 & 2 & \\ \hline & 1 & -2 & -1 & 0 & \end{array}$$

$\longrightarrow$ zero

$p(x) = (x-5)(x^3 - 5x - 2)$

$\Rightarrow p(x) = (x-5)(x+2)(x^2 - 2x - 1)$

$\Delta = 4 - 4(-1) = 8 =$

$\Rightarrow x = \dfrac{+2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

$SS = \{-2, 5, 1 \pm \sqrt{2}\}$

Summary.

1) If all zeros are rational, we can find them all.

2) If a poly is of deg 3 & 1 zero is rational we can find them all.

3) If deg $p(x) = 4$ & 2 zeros are rational $\Rightarrow$ we can find all zeros.

4) If deg $p(x) = 3$ or 4 & one complex zero is known $\Rightarrow$ we can find all zeros

5) If $p(x) = n$ & $n-2$ are rational we can find them all.