25/ 15/ 9/

King Fahd University of Petroleum and Minerals
Prep-Year Math Program

Prep-Year Math II
MIDTERM EXAM
Semester I, Term 061
Tuesday, November 14, 2006
Net Time Allowed: 120 minutes

"Sources of Problems"

MASTER VERSION

1. If for the function $f(x) = -3\sin(\pi x - 2) + 5$, A is its amplitude, P is its period, M is its maximum value and m is its minimum vlaue, then $\dfrac{A+P}{M+m} =$

(a) $\dfrac{1}{2}$

(b) 2

(c) $\dfrac{1}{4}$

(d) 4

(e) 1

See examples 2 and 3 p. 514-515
3 and 4 p. 531

See problems 1 to 8 p. 535

$f(x) = a\sin(bx + c) + d$

$a = -3$, $b = \pi$ $c = -2$ $d = 5$

$\boxed{A = |a| = 3}$, $\boxed{P = \frac{2\pi}{b} = \frac{2\pi}{\pi} = 2}$

$\boxed{M = |a| + d = 3 + 5 = 8}$ $\boxed{m = -|a| + d = -3 + 5 = 2}$

$$\frac{A+P}{M+m} = \frac{3+2}{8+2} = \frac{5}{10} = \frac{1}{2}$$

2. If f^{-1} is the inverse of the function $f(x) = 2^{-x+1} - 3$, then $f^{-1}(x) =$

(a) $1 - \log_2(x+3)$

(b) $3 + \log_2(x-1)$

(c) $-1 + \log_2(x+3)$

(d) $3 + \log_2(x+3)$

(e) $-1 + \log_2(3-x)$

A direct application of the relation between the exponential and logarithmic functions as inverses of one another.

$y = 2^{-x+1} - 3$

$x = 2^{-y+1} - 3$

$x + 3 = 2^{-y+1}$

Change to log form

$-y + 1 = \log_2(x+3)$

$-y = -1 + \log_2(x+3)$

$\boxed{y = 1 - \log_2(x+3)}$

$\overset{\shortparallel}{f^{-1}(x)}$

3. Which one of the following statements is **TRUE** about the function $f(x) = -2|\ln x|$ and its graph?

(a) increases on $(0,1)$

(b) increases on $(1,\infty)$

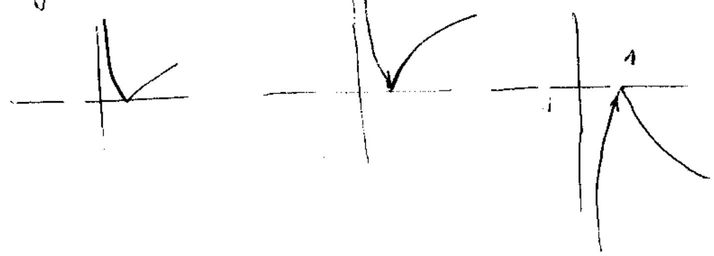
(c) decreases on $\left(\frac{1}{2}, 2\right)$

(d) f has no maximum value

(e) $x = 2$ is a vertical asymptote

See Problems 61, 62 and 67 P. 392

$y = |\ln(x)| \longrightarrow 2|\ln(x)| \longrightarrow -2|\ln(x)|$



4. The **domain** of $f(x) = \log_{x-1} x$ is

(a) $(1,2) \cup (2,\infty)$

(b) $(1,2)$

(c) $(1,\infty)$

(d) $(0,1) \cup (1,\infty)$

(e) $(2,\infty)$

To make sure that the base is a positive number different from 1 and $\log_b x$ is defined only for $x > 0$.

$x > 0 \quad \& \quad 0 < x-1 \quad \& \quad x-1 \neq 1$

$\Rightarrow \quad x > 1 \quad \& \quad x \neq 2$

$\Rightarrow \quad (1,2) \cup (2,\infty)$

5. Which one of the following statements is **TRUE** for all $x > 0$, $y > 0$, $b > 0$ and $b \neq 1$?

(a) $\log_b \sqrt{x} = \dfrac{\ln x}{2 \ln b}$

(b) $\log_b(x + y) = \log_b x + \log_b y$

(c) $\log_b x \cdot \log_b y = \log_b x + \log_b y$

(d) $\log_b \dfrac{x}{y} = \dfrac{\log_b x}{\log_b y}$, $y \neq 1$

(e) $\dfrac{\log_b x}{\log_b y} = \log_b x - \log_b y$, $y \neq 1$

See problems 31 to 40 p.404

b, c, d, e are clearly false.

a) $\log_b \sqrt{x} = \dfrac{\ln \sqrt{x}}{\ln b} = \boxed{\dfrac{1}{2} \dfrac{\ln x}{\ln b}}$

6. The value of $(\ln 10000)(\log \sqrt{e})(\log_3 \sqrt{5})(\log_5 9)$ is

(a) 2

(b) 8

(c) 1

(d) 4

(e) 6

See problems 41 and 42 p.404

$\ln 10^4 \dfrac{\ln \sqrt{e}}{\ln 10} \cdot \dfrac{\ln \sqrt{5}}{\ln 3} \cdot \dfrac{\ln 9}{\ln 5}$

$(4 \cancel{\ln 10}) \dfrac{(\frac{1}{2} \ln e)}{\cancel{\ln 10}} \cdot \dfrac{\cancel{\frac{1}{2}} \cancel{\ln 5}}{\cancel{\ln 3}} \cdot \dfrac{\cancel{2} \cancel{\ln 3}}{\cancel{\ln 5}}$

$2 \ln e = 2$

7. $\dfrac{\tan 155^\circ - \cot 35^\circ}{1 + \tan 155^\circ \cot 35^\circ} =$

(a) $-\tan 80^\circ$

(b) $\tan 20^\circ$

(c) $-\tan 75^\circ$

(d) $-\tan 25^\circ$

(e) $\tan 15^\circ$

An application of:

- $\cot \alpha = \tan(\frac{\pi}{2} - \alpha)$
- $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$.

$$\frac{\tan 155^\circ - \tan(90 - 35^\circ)}{1 + \tan 155 \tan(90-35)^\circ} = \frac{\tan(155^\circ) - \tan 55^\circ}{1 + \tan 155 \tan 55}$$

$$= \tan(100^\circ) = \boxed{-\tan(80^\circ)}$$

$\pi \to \tan \ominus$

$\theta' = 80^\circ$

8. If $\dfrac{10^x - 10^{-x}}{10^x + 10^{-x}} = \dfrac{1}{2}$, then $x =$

(a) $\log\sqrt{3}$

(b) $\log\sqrt[3]{2}$

(c) $\log 3$

(d) $\log 2$

(e) $\dfrac{3}{\log 2}$

See example 4 p.410

See problems 39 to 46 p. 415

$$2(10^x - 10^{-x}) = 10^x + 10^{-x}$$

$$2(10)^x - 2 \cdot 10^{-x} = 10^x + 10^{-x}$$

$$10^x - 3 \cdot 10^{-x} = 0$$

$\times 10^x$

$$(10^x)^2 - 3 = 0$$

$$(10^x)^2 = 3 \qquad 10^x = \pm\sqrt{3}$$

rejected

$$\boxed{x = \log\sqrt{3}}$$

9. The number of **intersection** points of the graphs of $y = \sin 2\pi x$ and $y = \cos 2\pi x$, over the interval $0 \le x \le \frac{3}{4}$, is

(a) 2

(b) 1

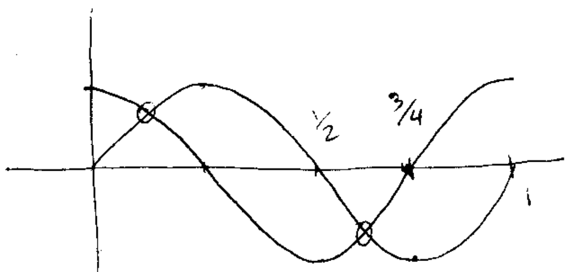
(c) 5

(d) 3

(e) 4

See problems 63 and 64 p. 519

Graph $P = \frac{2\pi}{2\pi} = 1$



2 intersection points.

10. If $x = \frac{3}{2}\sin\theta$, where $0 < \theta < \frac{\pi}{2}$, then the expression $\frac{12x^2}{(9-4x^2)^{3/2}}$ simplifies to

(a) $\tan^2\theta\sec\theta$

(b) $\tan^2\theta\sin\theta$

(c) $\tan^2\theta\cos\theta$

(d) $\cot^2\theta\sec\theta$

(e) $\cot^2\theta\sin\theta$

A direct application of the Fundamentamental Trigonometric Identities of Section 6.1

$$x^2 = \frac{9}{4}\sin^2\theta$$

$$\frac{12x^2}{(9-4x^2)^{3/2}} = \frac{3 \cdot 9 \cdot \sin^2\theta}{(9 - 9\sin^2\theta)^{3/2}} = \frac{27\sin^2\theta}{[9(1-\sin^2\theta)]^{3/2}}$$

$$= \frac{\cancel{27}\sin^2\theta}{\cancel{27}\cdot\cos^3\theta} = \frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\cos\theta} = \tan^2\theta\cdot\sec\theta$$

11. The expression $\dfrac{\sin^3 x - \cos^3 x}{\sin x - \cos x}$ simplifies to

(a) $1 + \sin x \cos x$

(b) $1 + 2\sin x \cos x$

(c) $1 - 2\sin x \cos x$

(d) $-1 - \sin x \cos x$

(e) $-1 + \sin x \cos x$

See problem 59 p. 561

$$\frac{\sin^3 x - \cos^3 x}{\sin x - \cos x} = \frac{(\sin x - \cos x)(\sin^2 + \sin x \cos x + \cos^2 x)}{(\sin x - \cos x)}$$

$$= \underbrace{\sin^2 x + \sin x \cos x + \cos^2 x}_{1}$$

$$= 1 + \sin x \cos x$$

12. If $y = 2\cot 2x$, then the number of vertical asymptotes over the interval $\left(-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$ is equal to

(a) 2

(b) 1

(c) 3

(d) 4

(e) 0

See example 3 p. 523

See problems 31, 32, 40, 45 and 48 p. 527

First find the eq^n of the V.A.

sin in denominator of cotangent

$$\Rightarrow \quad 2x = n\pi \xrightarrow[\text{for } x]{\text{solve}} x = n\frac{\pi}{2}$$

$$-\frac{\pi}{4} < n\frac{\pi}{2} < \frac{3\pi}{4}$$

$$-\frac{\pi}{2} < n < \frac{3\pi}{2}$$

$$-\frac{1}{2} < n < 3/2$$

n integer so

n = 0, 1

2 Vert. Asym

13. If $\sin\alpha = \frac{3}{5}$, α lies in second quadrant, and $\cos\beta = -\frac{5}{13}$, β lies in third quadrant, then $\sin\left(\frac{\pi}{2} - \alpha + \beta\right) =$

(a) $-\frac{16}{65}$

(b) $\frac{48}{65}$

(c) $-\frac{48}{65}$

(d) $\frac{12}{65}$

(e) $-\frac{36}{65}$

See problems 69 and 70 p.51,
then problems 39 to 48 p. 570-571

$\left(\frac{\pi}{2} - \alpha + \beta\right)$ reminds of $\frac{\pi}{2} - \alpha$ in cofunction identity

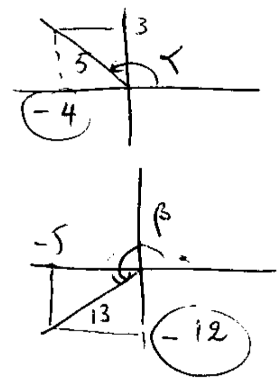
$\sin\left(\frac{\pi}{2} - (\alpha - \beta)\right) = \cos(\alpha - \beta)$

$= \cos\alpha \cos\beta + \sin\alpha \sin\beta.$

$= \left(\frac{-4}{5}\right)\left(\frac{-5}{13}\right) + \left(\frac{3}{5}\right)\left(\frac{-12}{13}\right)$

$= \frac{20 - 36}{65}$

$= \boxed{-\frac{16}{65}}$



14. If a wheel of radius 8 centimeters is rotating at 450 revolutions per minute, then the **linear speed** of a point on the edge of the wheel in centimeters per seconds is equal to

(a) 120π

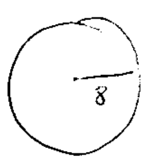
(b) 240π

(c) 110π

(d) 220π

(e) 230π

See example 8 p. 471
See problems 73 and 74 p. 473



$v = \omega \cdot r$

$= 450 \frac{\text{rev}}{\text{min}} \cdot 8 \text{ cm}$

$= \frac{\cancel{450}^{15} \cdot \cancel{2}\pi}{\cancel{60}} \cdot 8 \text{ cm/sec}$

$= 120\pi \text{ cm/sec}$

15. The line $y = 3$ intersects the graph of $y = -2\csc\dfrac{x}{3}$ over the interval $\left(-\dfrac{3\pi}{2}, 6\pi\right)$ at

(a) three points ✓

(b) one point

(c) two points

(d) four points

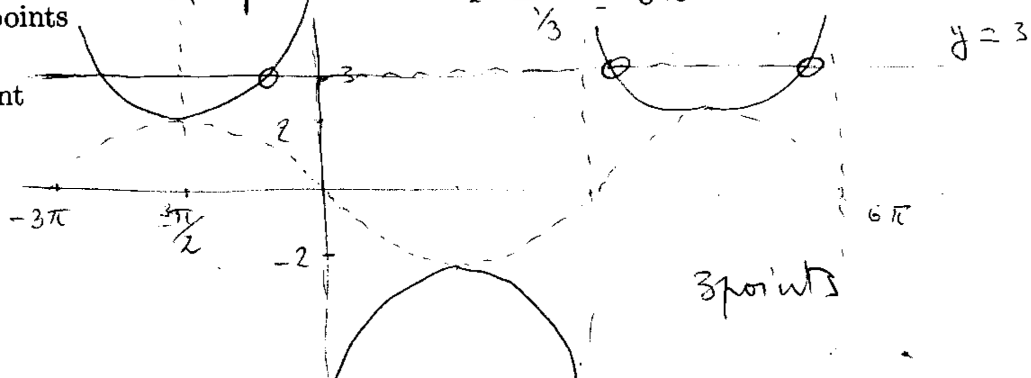
(e) no point

See example 4 p. 524

See problems 33, 34, 38, 41, and 44 p. 527

Graph

$P = \dfrac{2\pi}{1/3} = 6\pi$



16. The reference angle θ', in radians, of the angle $\theta = -1656°$ is equal to

(a) $\dfrac{\pi}{5}$ ✓

(b) $\dfrac{\pi}{7}$

(c) $\dfrac{\pi}{3}$

(d) $\dfrac{\pi}{6}$

(e) $\dfrac{\pi}{9}$

See example 3 p. 494

See problems 25 to 36 p. 497

$\theta = -1656°$ contains 4 neg rev → so add (4 + 1) rev

$-1656° + 4(360°) = -1656° + 1800$

$= 144° \in \text{II}$

ref angle: $180° - 144° = 36°$

$36° = 36° \times \dfrac{\pi}{180°}$ rad

$= \dfrac{\pi}{5}$ rad

17. The expression $\dfrac{\tan t}{1+\sec t}+\dfrac{1+\sec t}{\tan t}$ simplifies to

(a) $2\csc t$

(b) $2\sec t$

(c) $2\cot t$

(d) $2\tan t$

(e) $2\sin t$

See example 4 p. 558

See problems 49 to 52 p. 560

First, notice $(1+\sec t)(1-\sec t) = 1-\sec^2 t$ conjugate $= 1-\tan^2 t - 1 = -\tan^2 t$.

$$\frac{\tan t\,(1-\sec t)}{(1+\sec t)(1-\sec t)} + \frac{1+\sec t}{\tan t}$$

$$\frac{\cancel{\tan t}\,(1-\sec t)}{-\tan^{\cancel{2}} t} + \frac{(1+\sec t)}{\tan t}$$

$$\frac{-1+\sec t + 1+\sec t}{\tan t} = \frac{2\sec t}{\tan t} = 2\cdot\frac{1}{\cancel{\cos t}}\cdot\frac{\cancel{\cos t}}{\sin t} = \boxed{2\csc t}$$

18. If the terminal side of an angle θ lies on the line $3x+4y=0$, where $x>0$, then the value of $\cot\theta+\cos\theta$ is

(a) $-\dfrac{8}{15}$

(b) $-\dfrac{32}{15}$

(c) $\dfrac{32}{15}$

(d) $\dfrac{1}{15}$

(e) $-\dfrac{1}{5}$

See example 1 p. 491

See problems 1 to 8 p. 497

Choose $x=4 \Rightarrow 3(4)+4y=0 \Rightarrow 4y=-12$

$\boxed{y=-3}$

$\cot\theta = \dfrac{x}{y} = \dfrac{4}{-3} = -\dfrac{4}{3}$, $r=\sqrt{4^2+(-3)^2}=5$

$\cos\theta = \dfrac{x}{r} = \dfrac{4}{5}$

$$\cot\theta+\cos\theta = -\frac{4}{3}+\frac{4}{5} = -\frac{20}{15}+\frac{12}{15} = \boxed{-\frac{8}{15}}$$

19. Which one of the following is an **odd** function?

~~(a)~~ $f(x) = \dfrac{3\cos x}{x^2 \tan x + \csc x}$ → even → odd

See Problems 41 to 48 P. 508

(b) $f(x) = x^3 + \tan^2 x$ → neither

(c) $f(x) = \dfrac{1 + x\cos x}{\sin x + \tan x}$ neither

(d) $f(x) = \dfrac{x^2}{3 + \cos x}$ → e

(e) $f(x) = x^3 \csc x + 1$

e

20. The exact value of the expression

$$2\cos\left(-\frac{7\pi}{4}\right)\tan(240^\circ) - \sqrt{6}\csc\left(\frac{7\pi}{6}\right)$$

is

See problems 65 and 66 P. 498

~~(a)~~ $3\sqrt{6}$

(b) $2\sqrt{6}$

(c) $\sqrt{6}$

(d) $-\sqrt{6}$

(e) $-2\sqrt{6}$

$2\cos\left(\frac{7\pi}{4}\right)\tan 240^\circ - \sqrt{6}\,\frac{1}{\sin\left(\frac{7\pi}{6}\right)}$

IV, → cos ⊕, $\theta' = \frac{\pi}{4}$ III → sin ⊖ $\theta' = \frac{\pi}{6}$

$2\cos\frac{\pi}{4}\tan 60^\circ - \sqrt{6}\,\frac{1}{\left(-\sin\frac{\pi}{6}\right)}$

$= \cancel{2}\cdot\frac{\sqrt{2}}{\cancel{2}}\cdot\sqrt{3} - \frac{\sqrt{6}}{-\frac{1}{2}}$

$= \sqrt{6} + 2\sqrt{6} = 3\sqrt{6}$

21. The range R and the period P of the function $y = -\left|3\sin\frac{x}{2}\right|$ are given by [Hint: Sketch]

(a) $R = [-3, 0]$, $P = 2\pi$

(b) $R = [-3, 3]$, $P = 2\pi$

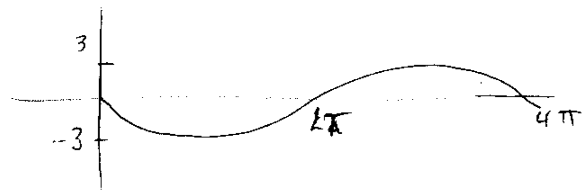
(c) $R = [-3, 0]$, $P = 4\pi$

(d) $R = [-3, 0]$, $P = \pi$

(e) $R = [-3, 0]$, $P = \frac{\pi}{2}$

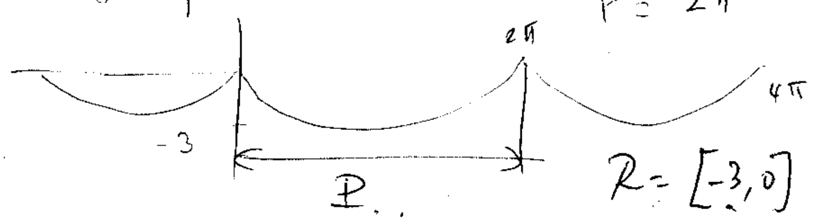
See problems 55 & 58 p. 527

$y = -3\sin\frac{x}{2}$ $P = 4\pi$.



$y = -\left|3\sin\frac{x}{2}\right|$

$P = 2\pi$



$R = [-3, 0]$

22. If L is the distance between the two points $P_1(\cos\theta, \sin\theta)$ and $P_2(\cos 2\theta, \sin 2\theta)$, then $L^2 =$

(a) $2 - 2\cos\theta$

(b) $2 + 2\sin\theta$

(c) $2 + 2\cos 3\theta$

(d) $3 - \cos\theta$

(e) $3 - \cos 3\theta$

See the distance between two points given p. 563 of section 6.2

$L = d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(\cos 2\theta - \cos\theta)^2 + (\sin 2\theta - \sin\theta)^2}$

$\Rightarrow L^2 =$

23. If, in the given figure, the length of AC is 10 cm, then the length of BD is

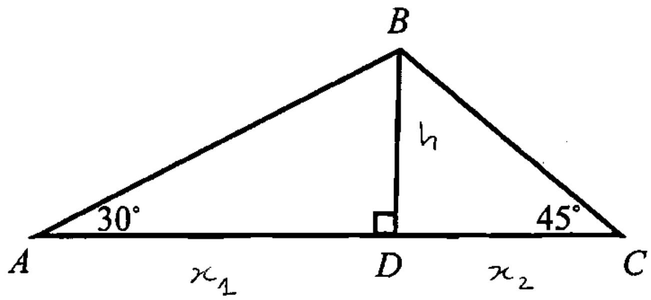
(a) $\dfrac{10}{1+\sqrt{3}}$ cm

(b) $\dfrac{10}{\sqrt{3}-1}$ cm

(c) $\dfrac{10\sqrt{3}}{1+\sqrt{3}}$ cm

(d) $\dfrac{10\sqrt{3}}{\sqrt{3}-1}$ cm

(e) $\dfrac{20}{1+\sqrt{3}}$ cm



An application of the tangent (or cotangent) function.

$x_1 + x_2 = 10$ $\Rightarrow \boxed{h = 10 - x_1}$

$\tan 45^\circ = 1 \Rightarrow \boxed{x_2 = h}$ $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{h}{10-h} \Rightarrow h\sqrt{3} = 10 - h$ $\boxed{h = \frac{10}{1+\sqrt{3}}}$

24. If the points $(-1,3)$ and $(2,81)$ lie on the graph of the exponential function $f(x) = b^{x+c}$, then $b+c=$

(a) 5

(b) 3

(c) -1

(d) 4

(e) -2

A point lies on the graph of a function $\Rightarrow$ the point satisfies the equation of the graph.

$3 = b^{-1+c} = \frac{b^c}{b} \Rightarrow b^c = 3b$

$81 = b^{2+c} = b^c \cdot b^2 = (3b)\,b^2 = 3b^3$

$\Rightarrow \frac{81}{3} = b^3 \Rightarrow 27 = b^3 \Rightarrow \boxed{b = 3}$

$b^c = 3^c = 3(3) = 9 \Rightarrow \boxed{c = 2}$ $\Rightarrow \boxed{(b+c) = 5}$

25. If the given graph represents the function $y = a\tan(bx+c)$ over the interval $(-1, 3)$, then the values of a, b and c are given by

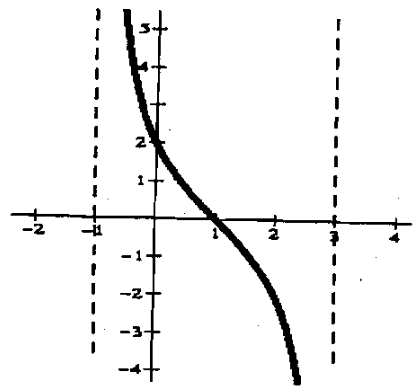
~~(a)~~ $a=-2,\ b=\frac{\pi}{4},$ and $c=-\frac{\pi}{4}$

(b) $a=2,\ b=\frac{\pi}{4},$ and $c=\frac{\pi}{4}$

(c) $a=-2,\ b=\frac{3\pi}{4},$ and $c=-\frac{\pi}{4}$

(d) $a=2,\ b=-\frac{\pi}{4},$ and $c=-\frac{\pi}{4}$

(e) $a=-2,\ b=4\pi,$ and $c=-4\pi$



See problem 50 p. 527

$P = 3 - (-1) = 4 = \frac{\pi}{b} \Rightarrow \boxed{b = \frac{\pi}{4}}$

Shape of $-\tan \Rightarrow \boxed{a = -2}$

PS = middle point of cycle $= 1 = -\frac{c}{b} = -\frac{c}{\left(\frac{\pi}{4}\right)}$

$\boxed{c = -\frac{\pi}{4}}$