King Fahd University of Petroleum and Minerals
Prep-Year Math Program

Exam II
Prep-Year Math II. Term (071)
December 9, 2007
Time Allowed: 90 Minutes

SEAT # ________

NAME: Key ID# ______________ SEC# ______

IMPORTANT INSTRUCTIONS:

DON'T OPEN THE EXAM BOOKLET (WAIT FOR FURTHER INSTRUCTIONS)

SHOW ALL YOUR WORK AND WRITE CLEAR STEPS

1) ALL TYPES OF CALCULATORS, PAGERS OR TELEPHONES ARE NOT ALLOWED DURING THE EXAMINATION.
2) WRITE YOUR NAME, ID NUMBER AND SECTION NUMBER.
3) USE ONLY PENCIL TO ANSWER THE QUESTIONS.
4) USE A GOOD ERASER, DON'T USE THE ERASER ATTACHED TO THE PENCIL.
5) WHEN YOU ARE TOLD, CHECK THAT THE EXAM PAPER HAS 11 QUESTIONS

1	2	3	4a	4b	5	6	7	8	9	10	11
4 pts	4 pts	10 pts	4 pts	4 pts	5 pts	10 pts	7 pts	3 pts	3 pts	3 pts	3 pts

TOTAL __________/60

1. (4 points) If $\sin\alpha = \dfrac{8}{17}$ and $90° < \alpha < 180°$, find the following:

(a) $\cos 2\alpha$.

$$= 1 - 2\sin^2\alpha$$
$$= 1 - 2\left(\frac{8}{17}\right)^2 \quad \Big\}\ 1\text{ pt}$$
$$= \frac{161}{289} \quad \Big\}\ 1\text{ pt}$$

(b) $\sin\dfrac{\alpha}{2}$. $45° < \frac{\alpha}{2} < 90°$, $\cos\alpha = \frac{-15}{17}$ --- $\frac{1}{2}$ pt

$$\sin\frac{\alpha}{2} = \sqrt{\frac{1+\frac{15}{17}}{2}} \quad \text{--- } \tfrac{1}{2}\text{ p}$$
$$= \frac{4\sqrt{17}}{17} \quad \text{--- } 1\text{ pt}$$

2. (4 points) Given $f(x) = 5\sqrt{2}\cos x - 5\sqrt{2}\sin x$,

(a) write $f(x)$ in the form $k\sin(x+\alpha)$, then

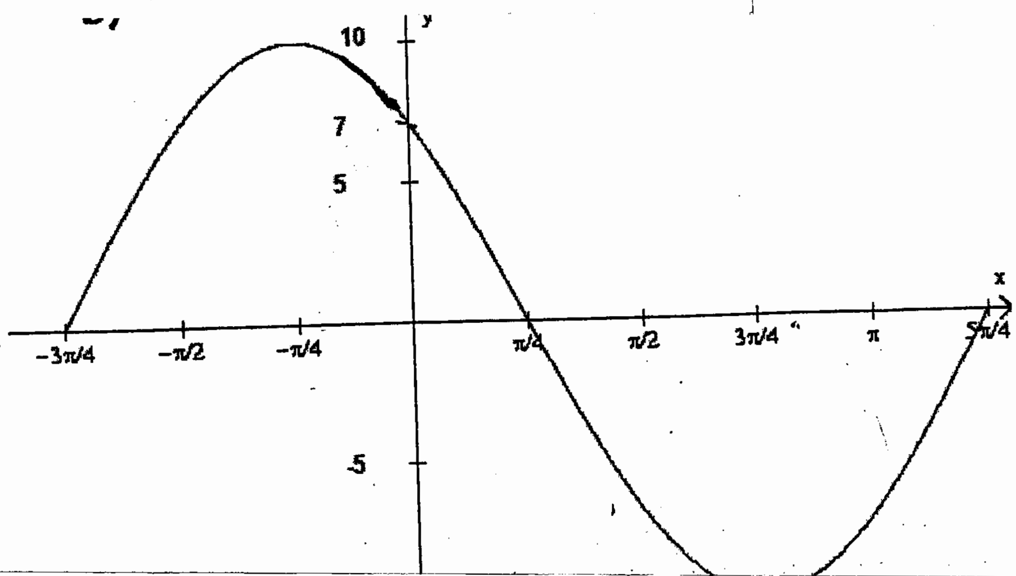
(b) graph f in the interval $\dfrac{-3\pi}{4} \le x \le \dfrac{5\pi}{4}$.

a) $k = \sqrt{50+50} = 10$ --- $\frac{1}{2}$ pt

$$\left.\begin{aligned}\cos\alpha &= -\frac{\sqrt{2}}{2}\\ \sin\alpha &= \frac{\sqrt{2}}{2}\end{aligned}\right\} \text{ -- } \tfrac{1}{2}\text{ p}$$

$$\left.\begin{aligned}\alpha &= \frac{3\pi}{4}\\ \therefore f(x) &= 10\sin\left(x+\frac{3\pi}{4}\right)\end{aligned}\right\} 1\text{ pt}$$

b)



2 pts.

3. (10 points) Whenever possible, find the value of each of the following:

(a) $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ --- 1 pt

(b) $\csc^{-1}\left(-\frac{1}{2}\right) =$ undefined --- 1 pt

(c) $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$ --- 1 pt

(d) $\sin^{-1}\left(\sin\frac{5\pi}{9}\right) = \sin^{-1}\left[\sin\left(\frac{4\pi}{9}\right)\right]$ --- 1 pt

$= \frac{4\pi}{9}$ --- 1 pt.

(e) $\cos\left(\sin^{-1}\frac{5}{13}\right) = \cos\left[\cos^{-1}\frac{12}{13}\right]$ -- 1 pt

$= \frac{12}{13}$ ---- 1 pt

(f) $\tan\left(\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{3}{5}\right)$.

$$\left.\begin{aligned} \cos^{-1}\tfrac{3}{5} &= \tan^{-1}\tfrac{4}{3} \\ \sin^{-1}\tfrac{3}{5} &= \tan^{-1}\tfrac{3}{4} \end{aligned}\right\} \text{1pt}$$

$\tan\left(\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{3}{5}\right) = \dfrac{\frac{4}{3} - \frac{3}{4}}{1 + \frac{4}{3}\cdot\frac{3}{4}} = \frac{7}{24}$ 1 pt

4. (a) (4 points) Solve $6\cos x\sin x - 3\cos x - 8\sin x + 4 = 0$, where $0 \le x < 2\pi$.

$$\left.\begin{aligned} 3\cos x[2\sin x - 1] - 4[2\sin x - 1] &= 0 \\ (2\sin x - 1)(3\cos x - 4) &= 0 \end{aligned}\right\} \text{— 1 pt}$$

$\sin x = \frac{1}{2}$ or $\cos x = \frac{4}{3}$ --- 1 pt

impossible

$x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

$S.S = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$ 1 pt

(b) (4 points) Solve $\sin 2x - \sin x = 0$.

$2\sin x \cos x - \sin x = 0$ $\frac{1}{2}$ pt

$\sin x(2\cos x - 1) = 0$ -- $\frac{1}{2}$ pt

$\sin x = 0$ — $\frac{1}{2}$pt OR $\cos x = \frac{1}{2}$ -- $\frac{1}{2}$pt

$\Rightarrow x = n\pi$

1 pt

$$x = \begin{cases} \frac{\pi}{3} + 2n\pi \\ \frac{5\pi}{3} + 2n\pi \end{cases}$$ 1.5 pt

n is any integer

5. (5 points) Given the equation of the parabola $2x - y^2 - 6y + 1 = 0$,

(a) write the equation in standard form

$y^2 + 6y = 2x + 1 \Rightarrow y^2 + 6y + 9 = 2x + 10$ 1 – pt

$(y+3)^2 = 2(x+5)$ -- 1 pt

(b) find the equation of the directrix, and the coordinates of the focus of the parabola.

$4p = 2 \Rightarrow p = \frac{1}{2}$ --- $\frac{1}{2}$p

Vertex $(h, k) = (-5, -3)$ -- $\frac{1}{2}$p

Focus $(h+p, k) = (-5 + \frac{1}{2}, -3)$

$= (\frac{-9}{2}, -3)$ --- 1 pt.

Directrix : $x = h - p$ — 1 pt

$x = \frac{-11}{2}$

6. (10 points) Given the vectors $\mathbf{u} = \langle -2, 2\sqrt{3} \rangle$, and $\mathbf{v} = -2\sqrt{3}\,\mathbf{i} + 2\mathbf{j}$, find the following:

(a) $\|\mathbf{u}\|$. $= \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+12} = 4$

½ pt ½ pt

(b) The unit vector $\mathbf{e}$ in the opposite direction of $\mathbf{u}$.

$e = -\dfrac{\vec{u}}{\|\vec{u}\|} \underbrace{= -}_{1\text{ pt}} \dfrac{\langle -2, 2\sqrt{3} \rangle}{4} = \langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \rangle$ --- 1 pt

(c) The direction angle of $\mathbf{v}$.

1 pt $\left\{ \tan\theta = \dfrac{b}{a} \Rightarrow \tan\theta = \dfrac{2}{-2\sqrt{3}} = \dfrac{-1}{\sqrt{3}} \right.$, $\theta \in Q\,II$

$\theta = 180^\circ - 30^\circ = 150^\circ = \dfrac{5\pi}{6}$

(d) The vector $2\mathbf{u} - \sqrt{3}\,\mathbf{v}$.

$= 2\langle -2, 2\sqrt{3} \rangle - \sqrt{3}\langle -2\sqrt{3}, 2 \rangle$ — ½

$= \langle -4, 4\sqrt{3} \rangle + \langle 6, -2\sqrt{3} \rangle$ ½ p

$= \langle 2, 2\sqrt{3} \rangle$ — 1 pt

(e) The smallest nonnegative angle between $\mathbf{u}$ and $\mathbf{v}$. let α = the angle

$\cos\alpha = \dfrac{u \cdot v}{\|u\|\,\|v\|} = \dfrac{\langle -2, 2\sqrt{3} \rangle \langle -2\sqrt{3}, 2 \rangle}{(4)(4)}$ — 1 pt

$= \dfrac{4\sqrt{3} + 4\sqrt{3}}{16} = \dfrac{\sqrt{3}}{2}$ ½ pt

$\Rightarrow \alpha = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6} \; (= 30^\circ)$ ½ pt

(f) $\text{Proj}_{\mathbf{v}}\,\mathbf{u}$.

$= \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}$ ½ p $= \dfrac{8\sqrt{3}}{4}$ ½ p

$= 2\sqrt{3}$

equation is: $9x^2 + 4y^2 + 18x - 24y + 9 = 0$.

(a) Write the equation in standard form

$$9(x^2+2x+1) + 4(y^2-6y+9) = -9+9+36$$

$$9(x+1)^2 + 4(y-3)^2 = 36$$

$$\frac{(x+1)^2}{4} + \frac{(y-3)^2}{9} = 1$$

(2 pts.) $a^2 = 9 \Rightarrow a = 3$

$b^2 = 4 \Rightarrow b = 2$

(b) Find the coordinates of the center.

$(h,k) = (-1,3)$ (1 pt.)

(c) Find the coordinates of the endpoints of the major axis.

$V_1 = (h, a+k) = (-1, 6)$ (1 pt.)

$V_2 = (h, -a+k) = (-1, 0)$

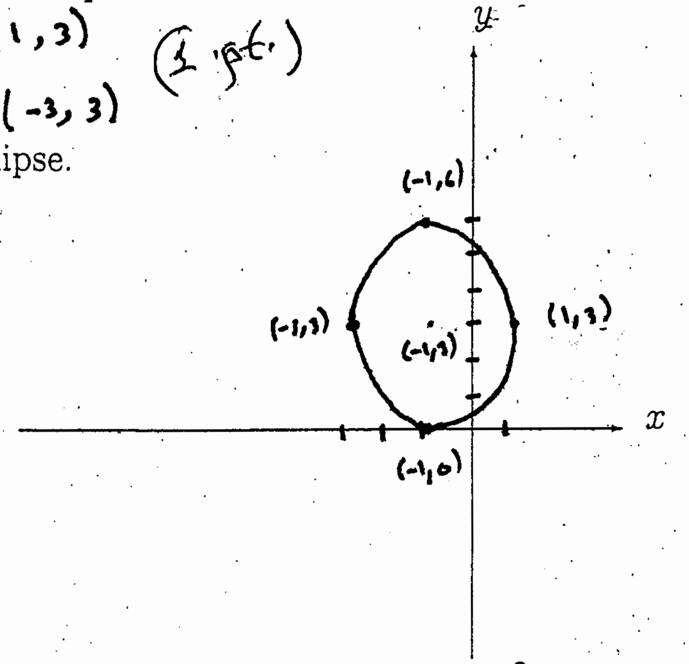
(d) Find the coordinates of the endpoints of the minor axis.

$P_1 = (b+h, k) = (1,3)$ (1 pt.)

$P_2 = (-b+h, k) = (-3, 3)$

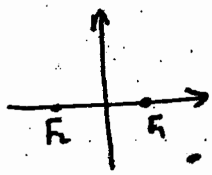
(e) Sketch the graph of the ellipse.

(2 pts.)



8. (3 points) Find the equation of the ellipse with eccentricity $\frac{2}{3}$, foci at $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$.

$e = \frac{c}{a} = \frac{2}{3}$. (1) (1 pt.)



$2c = d(F_1, F_2) \Rightarrow 2c = 2\sqrt{3} \Rightarrow c = \sqrt{3}$

Center $= (0,0)$ (The Midpoint between F_1 and F_2)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

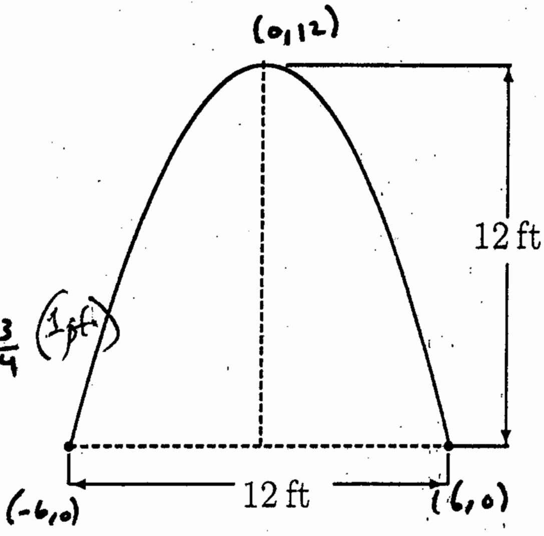
From (1) $2a = 3c \Rightarrow a = \frac{3}{2}c$

$\Rightarrow a = \frac{3\sqrt{3}}{2}$. $\Rightarrow a^2 = \frac{27}{4}$ 1 pt.

$\Rightarrow b^2 = a^2 - c^2 = \frac{27}{4} - 3 = \frac{15}{4}$.

Eqn: (1 pt.) $$\frac{x^2}{\frac{27}{4}} + \frac{y^2}{\frac{15}{4}} = 1$$

9. (3 points) A piece of wire is to be bent to form a part of a parabola as shown in the adjacent figure. Find the distance between the focus and the vertex of the parabola.



$(h, k) = (0, 12)$

Eqn: $x^2 = 4p(y-12)$ (1 pt.)

using the point $(6, 0)$

$\Rightarrow 36 = -48p \Rightarrow p = -\frac{36}{48} = -\frac{3}{4}$ (1 pt.)

$\therefore$ distance $= |p| = \frac{3}{4}$. (1 pt.)

10. (3 points) Calculate $(\sin 7.5°)^2$.

$$(\sin 7.5°)^2 = \frac{1-\cos 15°}{2} \;(1\text{ pt.}) = \frac{1-\sqrt{\frac{1+\cos 30°}{2}}}{2} \;(1\text{ pt.}) = \frac{1-\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}}{2} = \frac{2-\sqrt{2+\sqrt{3}}}{4} \;(1\text{ pt.})$$

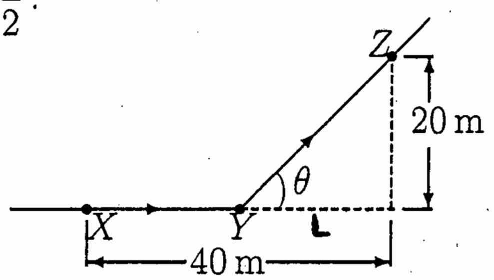
or $\cos 15° = \cos(45°-30°) = \cos 45° \cos 30° + \sin 45 \sin 30$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$

$= \frac{\sqrt{6}+\sqrt{2}}{4} \Rightarrow (\sin 7.5°)^2 = \frac{1-\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)}{2}$

$= \frac{4-\sqrt{6}-\sqrt{2}}{8}$

11. (3 points) A bus is passing by three bus stations $X, Y,$ and Z as shown in the figure. Show that the total distance d traveled by the bus from X to Z is given by $d = 40 + 20\tan\frac{\theta}{2}$.



$\|\vec{YZ}\| \sin\theta = 20$

$\Rightarrow \|\vec{YZ}\| = \frac{20}{\sin\theta}$ (1 pt.)

$L = \|\vec{YZ}\| \cos\theta \Rightarrow \|\vec{XY}\| = 40 - L$

$= 40 - \frac{20\cos\theta}{\sin\theta}$ (1 pt)

$\therefore d = \|\vec{XY}\| + \|\vec{YZ}\|$ (1 pt.)

$= 40 - \frac{20\cos\theta}{\sin\theta} + \frac{20}{\sin\theta}$

$= 40 + 20\left(\frac{1-\cos\theta}{\sin\theta}\right) = 40 + 20\tan\frac{\theta}{2}.$