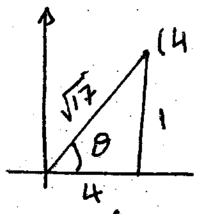
1) Find the exact values of each of the following expressions.

a) (2-points) $\csc(\tan^{-1}\frac{1}{4})$.

Let $\theta = \tan^{-1}(\frac{1}{4})$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ ---(0.5 pt)

$\Rightarrow \tan\theta = \frac{1}{4}$ ----------

$\Rightarrow \csc\theta = \sqrt{17}$ ----------(1 pt)



(0.5 p

b) (1-point) $\sin(\frac{1}{2}\sin^{-1} 2)$ is undefined ---(1 pt)

Any alternative word like

ϕ, impossible, rejected is acceptable.

2) (3-points) Given the vectors $u = \langle 12, -6 \rangle$ and $v = 4i - j$, find the magnitude and the degree measure of the direction angle of the vector $w = \frac{1}{2}u - 3v$.

Wrong calculations -1.5

$w = \frac{1}{2} \cdot \langle 12, -6 \rangle - 3 \cdot \langle 4, -1 \rangle$

$= \langle 6, -3 \rangle + \langle -12, 3 \rangle = \langle -6, 0 \rangle$ --(0.5 pt + 0.5

$\|w\| = \sqrt{(-6)^2 + 0^2} = \sqrt{6^2} = 6$ ------(1 pt)

$\tan\theta = \frac{0}{-6} = 0$

$\theta = 180^\circ$ or π --(1 pt)

OR

$\left.\begin{array}{l} -6 = \|w\|\cos\theta \Rightarrow -1 = \cos\theta \\ 0 = \|w\|\sin\theta \Rightarrow 0 = \sin\theta \end{array}\right\} \Rightarrow \theta.$

alternative

3) (3-points) If the horizontal and vertical components of a vector v are respectively 5 and -12, find the **unit vector** in the opposite direction of v.

$\|v\| = \sqrt{5^2 + (-12)^2} = 13$ ----------(1 pt)

unit vector in the direction of v is

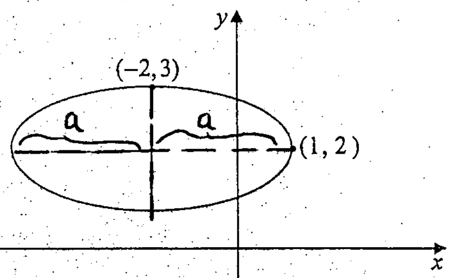
$\langle \frac{5}{13}, \frac{-12}{13} \rangle$ ------(1 pt)

unit ~~vector~~ in the opposite direction

of v is $\langle -\frac{5}{13}, \frac{12}{13} \rangle$ ------(1 pt)

- if the student starts with $v = \langle 5, -12 \rangle$, then he will given (0.5 pt)
- If $\|v\| = \sqrt{169}$ is not simplified, then (−0.5 pt)

4) (7-points) Shown in the figure is the ellipse with one vertex $(1, 2)$ and one endpoint $(-2, 3)$ of the minor axis given. Answer the following questions about this ellipse.



a) Find the center.

(1 pt) - - $(h, k) = (-2, 2)$

b) Find the other vertex.

$a = 1-(-2) = 3$

(1 pt) - - - the othe vertex is $(-5, 2)$

c) Find the lengths of the major and the minor axes.

(1 pt) — the length of the major axis is $2a = 2\cdot 3 = 6$

" " " " minor " " $2b = 2\cdot 1 = 2$

d) Find the foci.

(1 pt) — $c = \sqrt{a^2-b^2} = \sqrt{9-1} = \sqrt{8} = 2\sqrt{2}$

(1 pt) — Foci $= (h+c, k) = (-2+2\sqrt{2}, 2)$

(1 pt) — $(h-c, k) = (-2+2\sqrt{2}, 2)$

e) Find the eccentricity.

(1 pt) — $e = \dfrac{c}{a} = \dfrac{2\sqrt{2}}{3}$

f) Write the equation of the ellipse in standard form.

(1 pt) — $\dfrac{(x+2)^2}{9} + \dfrac{(y-2)^2}{1} = 1$

5) (3-points) Given $proj_w v = \frac{3}{5}$ and $\|v\| = \frac{6}{5}$, find the radian measure of the smallest positive angle between the vectors v and w.

$proj_w v = \|w\| \cos\theta$ - - - - - - (0.5 pt)

$\Rightarrow \frac{3}{5} = \frac{6}{5}\cos\theta$ - - - - (0.5 pt)

$\Rightarrow \cos\theta = \dfrac{\frac{3}{5}}{\frac{6}{5}} = \frac{1}{2}$ - - - (0.5 pt)

$\Rightarrow \theta = \frac{\pi}{3}$ — — — — (1 pt) if $(x-h)^2=4p(y-k)$, then (-2 pts)

6) (3-points) Find the equation, in standard form, of the parabola that has vertex $(1,2)$, has its axis of symmetry parallel to the x-axis, and passes through the point $(3,6)$.

The parabola opens to the right, thus the equation is $(y-k)^2=4p(x-h)$

(1 pt) ---- $\Rightarrow (y-2)^2=4p(x-1)$

the point $(3,6)$ Lies on the graph

(1 pt) -- $\Rightarrow (6-2)^2=4p(3-1) \Rightarrow 16=8p \Rightarrow p=2$

(1 pt) -- Thus, the equation is $(y-2)^2=8(x-1)$

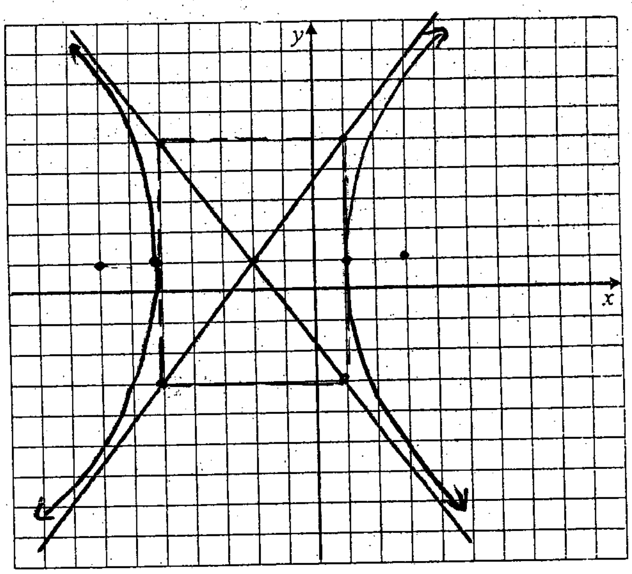
7) (8-points) Answer the following questions about the hyperbola given by the equation

$$\frac{(x+2)^2}{9}-\frac{(y-1)^2}{16}=1$$

$h=-2,\ k=1$
$a=3,\ b=4$
$c=5$

a) Find the center.

(1 pt) -- $(-2,1)$



b) Find the vertices.

(1 pt) — $(1,1)$ & $(-5,1)$

c) Find the foci.

(1 pt) -- $(3,1)$ & $(-7,1)$

d) Find the eccentricity.

(1 pt) -- $e=\frac{c}{a}=\frac{5}{3}$

e) Find the asymptotes.

(1 pt) -- $y-1=\pm\frac{4}{3}(x+2)$

f) Find the length of the Transverse axis.

(1 pt) -- $=2a=6$

g) Sketch the graph.

(2 pts)

8) (4-points) Solve the following system of equations by the elimination method.

$$2\sqrt{2}x+3\sqrt{5}y=7 \quad ---(Eq\,1)$$
$$3\sqrt{2}x-\sqrt{5}y=-17 \quad ---(Eq\,2)$$

$$\left.\begin{array}{l} -6\sqrt{2}x-9\sqrt{5}y=-21 \\ 6\sqrt{2}x-2\sqrt{5}y=-34 \end{array}\right\} ----(1\ pt)$$

$$\Rightarrow -11\sqrt{5}y=-55 \quad ----(0.5\ pt)$$

$$\Rightarrow y=\frac{55}{11\sqrt{5}}=\frac{5}{\sqrt{5}}=\sqrt{5} \quad ---(1\ pt)$$

Substitute in Eq①

$$2\sqrt{2}x+3\sqrt{5}\cdot\sqrt{5}=7$$

$$\Rightarrow 2\sqrt{2}x=-8 \Rightarrow x=\frac{-8}{2\sqrt{2}}=\frac{-4}{\sqrt{2}}=-2\sqrt{2} \ -(1\ pt)$$

$$\Rightarrow S.S.=\{(-2\sqrt{2},\sqrt{5})\} \quad ----(0.5\ pt)$$

(if solve by substitution, then (−1 pt.)

9) (5-points) Solve the trigonometric equation:

$$3+2\cos^2 x=2-\sqrt{3}\sin x \ , \quad 0\le x<2\pi$$

$$\Rightarrow 2\cos^2 x+\sqrt{3}\sin x+1=0$$

$$\Rightarrow 2(1-\sin^2 x)+\sqrt{3}\sin x+1=0 \quad ---(1\ pt)$$

$$\Rightarrow 2-2\sin^2 x+\sqrt{3}\sin x+1=0$$

$$2\sin^2 x-\sqrt{3}\sin x-3=0 \quad --(1\ pt)$$

$$\Rightarrow (2\sin x+\sqrt{3})(\sin x-\sqrt{3})=0$$

$$\Rightarrow \sin x=-\frac{\sqrt{3}}{2} \text{ or } \sin x=\sqrt{3}$$ (1 pt)

rejected

$$\Rightarrow x=\pi+\frac{\pi}{3},\ 2\pi-\frac{\pi}{3} \quad (1\ pt)$$

$$=\frac{4\pi}{3},\ \frac{5\pi}{3}$$ (1 pt)

OR

Alternative

$$\sin x=\frac{\sqrt{3}\pm\sqrt{3-4(2)(-3)}}{4}$$

$$=\frac{\sqrt{3}\pm\sqrt{27}}{4}=\frac{\sqrt{3}\pm3\sqrt{3}}{4}$$

$$\Rightarrow \sin x=\frac{\sqrt{3}+3\sqrt{3}}{4} \text{ or } \frac{\sqrt{3}-3\sqrt{3}}{4}$$

rejected

$$=\sqrt{3} \text{ or } -\frac{\sqrt{3}}{2} \ --(1$$

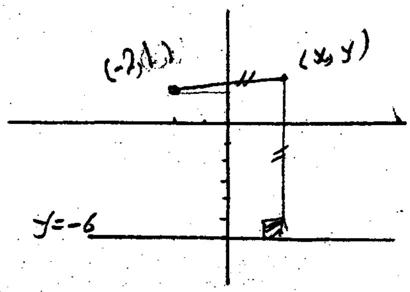
$$\Rightarrow x=\pi+\frac{\pi}{3} \text{ or } 2\pi-\frac{\pi}{3}$$

$$=\frac{4\pi}{3},\ \frac{5\pi}{3}$$ (1 pt)

10) (3-points) Find an equation of the set of points that are equally distant from the line $y=-6$ and the point $(-2,1)$.

(2 pt) — $\sqrt{(y-1)^2+(x+2)^2} = y+6$

(1 pt)
$\Rightarrow (y-1)^2+(x+2)^2=(y+6)^2$
$\Rightarrow y^2-2y+1+(x+2)^2 = y^2+12y+36$
$\Rightarrow (x+2)^2 = 14y+35$



11) (4-points) If P is any point on the ellipse given by $9x^2+4y^2-54x+32y+109=0$, then find the sum of the distances from P to the foci of the ellipse.

(1 pt) — $9(x^2-6x)+4(y^2+8y)=-109$
$9(x^2-6x+9)+4(y^2+8y+16)=-109+81+64$
(1 pt) — $9(x-3)^2+4(y+4)^2=36$
$\Rightarrow \dfrac{(x-3)^2}{4}+\dfrac{(y+4)^2}{9}=1$

(2 pts) — $a=3 \Rightarrow 2a=6$ = the sum of the distance from P to the foci

12) (4-points) Find, in interval notation, the **domain** and the **range** of the function

$$f(x)=-\frac{\pi}{3}+\frac{1}{2}\cos^{-1}\frac{1}{2}(x-1).$$

(1 pt) — $-1 \le \frac{1}{2}(x-1) \le 1 \Rightarrow -2 \le x-1 \le 2 \Rightarrow -1 \le x \le 3$ --- (0.5
$\Rightarrow$ The domain is $[-1,3]$ ---- (0.5 pt)

(1 pt)
$0 \le \cos^{-1}\frac{1}{2}(x-1) \le \pi \Rightarrow 0 \le \frac{1}{2}\cos^{-1}\frac{1}{2}(x-1) \le \frac{\pi}{2}$ --- (0.5 pt)
$\Rightarrow -\frac{\pi}{3}+0 \le -\frac{\pi}{3}+\frac{1}{2}\cos^{-1}\frac{1}{2}(x-1) \le -\frac{\pi}{3}+\frac{\pi}{2}$ --- (0.5 pt)
$\Rightarrow$ The range is $[-\frac{\pi}{3}, \frac{\pi}{6}]$

13) (5-points) Solve the trigonometric equation:

Wrong factoring (-1) (1 pt)

$\cos 6x - \cos 3x = 0$, $0^\circ \le x < 180^\circ$

$\cos 2(3x) - \cos 3x = 0 \Rightarrow 2\cos^2 3x - \cos 3x - 1 = 0$, $0 \le 3x \le 540^\circ$

$\Rightarrow (2\cos 3x + 1)\cdot(\cos 3x - 1) = 0$ --- (1 pt)

$\Rightarrow \cos 3x = -\frac{1}{2}$ or $\cos 3x = 1$ --- (0.5 pt)

$\Rightarrow 3x = \begin{cases} \frac{2\pi}{3} + 2k\pi = \frac{2\pi + 6k\pi}{3} \\ \frac{4\pi}{3} + 2k\pi = \frac{4\pi + 6k\pi}{3} \end{cases}$ 0.5 pt

$\Rightarrow x = \begin{cases} \frac{2\pi + 6k\pi}{9} \\ \frac{4\pi + 6k\pi}{9} \end{cases}$ 0.5 pt

0.5 pt $k = 0 \Rightarrow x = \begin{cases} \frac{2\pi}{9} = 40^\circ \\ \frac{4\pi}{9} = 80^\circ \end{cases}$

$k = 1 \Rightarrow x = \frac{8\pi}{9} = 160^\circ$

$\Rightarrow 3x = 0 + 2k\pi = 2k\pi$, $\Rightarrow x = \frac{2k\pi}{3}$ (0.5 pt)

$k=0 \Rightarrow x = 0 = 0^\circ$, $k=1 \Rightarrow x = \frac{2\pi}{3} = 120^\circ$ (0.5 pt)

OR

$\cos 3x = 1$, $\Rightarrow 3x = 0, 360^\circ$, $\Rightarrow x = 0, 120^\circ$ (1 pt)

$\cos 3x = -\frac{1}{2}$, $\Rightarrow 3x = 120^\circ, 240^\circ, 480^\circ$, $\Rightarrow x = 40^\circ, 80^\circ, 160^\circ$ (1 p

$\Rightarrow$ S.S. $= \{0^\circ, 40^\circ, 80^\circ, 120^\circ, 160^\circ\}$

14) (5-points) The graph of the function $f(x) = A + B\sin^{-1}(x-3)$ passes through the points $\left(\frac{7}{2}, \frac{\pi}{3}\right)$ and $\left(4, \frac{\pi}{2}\right)$. Find the exact values of A and B.

$A + B\sin^{-1}\left(\frac{7}{2} - 3\right) = \frac{\pi}{3}$ --(0.5 pt)

$A + B\sin^{-1}(4 - 3) = \frac{\pi}{2}$ --(0.5 pt)

$\Rightarrow A + B\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$, $A + B\sin^{-1}(1) = \frac{\pi}{2}$

$\Rightarrow A + B\cdot\frac{\pi}{6} = \frac{\pi}{3}$

$\Rightarrow A + B\cdot\frac{\pi}{2} = \frac{\pi}{2}$ --(1.5 pt)

$\Rightarrow -A - B\cdot\frac{\pi}{6} = -\frac{\pi}{3}$, $A + B\cdot\frac{\pi}{2} = \frac{\pi}{2}$

$B\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{\pi}{2} - \frac{\pi}{3}$ (1

$\Rightarrow B\cdot\frac{\pi}{3} = \frac{\pi}{6} \Rightarrow B = \frac{1}{2}$ --(0.5 pt)

& $A = \frac{\pi}{2} - B\cdot\frac{\pi}{2} = \frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$ ---(0.5 pt)