1. (7-points) Given the parabola $4x + y^2 - 6y + 1 = 0$:

(a) Write its equation in standard form.

$$\Rightarrow (y-3)^2 = -4x - 1 + 9$$
$$= -4x + 8 \quad \text{--- 1 pt}$$
$$\Rightarrow (y-3)^2 = -4(x-2) \quad \text{--- 1 pt}$$

(b) Find the coordinates of its focus.

Vertex $(h, k) \equiv (2, 3)$, $p = -1$

$\Rightarrow$ focus $(h+p, k) \equiv (1, 3)$ --- 1 pt

(c) Find the equation of its directrix.

Directrix $x = h - p$ --- 1 pt

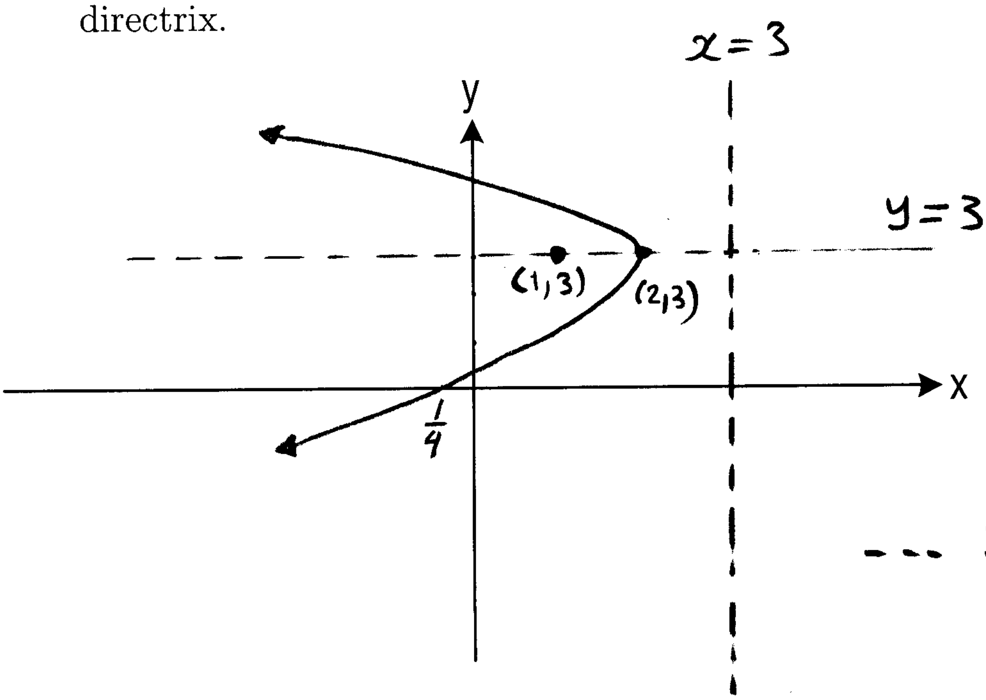
$\Rightarrow x = 3$

(d) Find the equation of its axis of symmetry.

$y = k$ --- 1 pt

$\Rightarrow y = 3$

(e) Sketch its graph showing its focus, vertex, axis of symmetry and directrix.



--- 2 pts

2. (4-points) Solve $2\sin^2 x + 3\sin x + 1 = 0$, where $0^\circ \le x < 360^\circ$.

$\Rightarrow$ $(2\sin x + 1)(\sin x + 1) = 0$ --- 1 pt

$\Rightarrow$ $\sin x = \frac{-1}{2}$ or $\sin x = -1$ --- 1 pt

$\Rightarrow$ $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{3\pi}{2}$ --- 2 pts

3. (4-points) If θ is the measure of the smallest positive angle between the vectors $\mathbf{v} = \langle 3, -4 \rangle$ and $\mathbf{w} = \langle 12, 5 \rangle$, find the exact value of $\cos\theta$.

$v \cdot w = (3)(12) + (-4)(5) = 36 - 20$

$= 16$ --- 1 pt

$\|v\| = \sqrt{9+16} = \sqrt{25} = 5$ --- 1 pt

$\|w\| = \sqrt{144+25} = \sqrt{169} = 13$ --- 1 pt

$\Rightarrow$ $\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|}$

$= \frac{16}{65}$ --- 1 pt.

4. (4-points) Find the equation in standard form of the ellipse that has center $(2,4)$, major axis parallel to the y-axis and of length 10, and passes through the point $(3,3)$.

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

with $h=2$, $k=4$, $a=5$ ... 1 pt

$$\Rightarrow \frac{(x-2)^2}{b^2} + \frac{(y-4)^2}{25} = 1 \quad \text{... 1 pt}$$

Now $(3,3)$ is on the ellipse $\Rightarrow$

$$\frac{1}{b^2} + \frac{1}{25} = 1$$

$$\Rightarrow \frac{1}{b^2} = \frac{24}{25} \Rightarrow b^2 = \frac{25}{24} \quad \text{... 1 pt}$$

$$\Rightarrow \frac{(x-2)^2}{\frac{25}{24}} + \frac{(y-4)^2}{25} = 1 \quad \text{... 1 pt}$$

5. (4-points) Find the exact value of $\cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}\left(\tan\frac{5\pi}{6}\right)$.

$$0 \leq \cos^{-1}\left(-\tfrac{1}{2}\right) \leq \pi \Rightarrow \cos^{-1}\left(-\tfrac{1}{2}\right) = \frac{2\pi}{3} \quad \text{... 1 pt}$$

$$\tan\frac{5\pi}{6} = \tan\left(-\frac{\pi}{6}\right) \quad \text{... 1 pt}$$

$$\Rightarrow \tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \tan^{-1}\left(\tan -\frac{\pi}{6}\right) = -\frac{\pi}{6}$$

... 1 pt

$$\Rightarrow \cos^{-1}\left(-\tfrac{1}{2}\right) + \tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi}{6} = \frac{\pi}{2} \quad \text{... 1 pt}$$

6. (4-points) Find all values of x for which $\sec 2x - 1 = \tan^2 2x$.

$\Rightarrow \quad \sec 2x = 1 + \tan^2 2x = \sec^2 2x$

$\Rightarrow \quad \sec^2 2x - \sec 2x = 0$

$\Rightarrow \quad \sec 2x\,(\sec 2x - 1) = 0$ --- 1 pt

$\Rightarrow \quad \sec 2x = 0$ rejected --- 1 pt

OR $\quad \sec 2x = 1 \Rightarrow 2x = 0, \pm 2\pi, \pm 4\pi, \ldots$

--- 1 pt

$\Rightarrow \quad x = k\pi$, k is any integer --- 1 pt

7. (4-points) Find the eccentricity, the coordinates of the vertices and the foci of the ellipse $\dfrac{(x-2)^2}{25} + \dfrac{(y+2)^2}{9} = 1$.

Center: $(h, k) \equiv (2, -2)$

$a = 5, \quad b = 3$

and $c = \sqrt{a^2 - b^2} = \sqrt{25 - 9} = \sqrt{16} = 4$ --- 1 pt

$\Rightarrow$ eccentricity $e = \dfrac{c}{a} = \dfrac{4}{5}$ --- 1 pt

Vertices: $(h+a, k) \equiv (7, -2)$

$(h-a, k) \equiv (-3, -2)$ --- 1 pt

foci: $(h+c, k) = (6, -2)$

$(h-c, k) = (-2, -2)$ --- 1 pt

8. Given $\mathbf{v} = \langle 2, 4 \rangle$ and $\mathbf{w} = -2\mathbf{i} + 8\mathbf{j}$, find:

(a) (4-points) $\left\| \frac{2}{3}\mathbf{v} + \frac{1}{6}\mathbf{w} \right\|$.

$\frac{2}{3} v = < \frac{4}{3}, \frac{8}{3} >$ ... 1 pt

$\frac{1}{6} w = < -\frac{1}{3}, \frac{4}{3} >$ ... 1 pt

$\Rightarrow \frac{2}{3} v + \frac{1}{6} w = < 1, 4 >$ ... 1 pt

$\| \frac{2}{3} v + \frac{1}{6} w \| = \sqrt{1+16} = \sqrt{17}$ ... 1 pt

(b) (4-points) $\text{Proj}_{\mathbf{w}}\mathbf{v}$. $= \frac{v \cdot w}{\|w\|}$ ... 1 pt

$v \cdot w = -4 + 32 = 28$ ... 1 pt

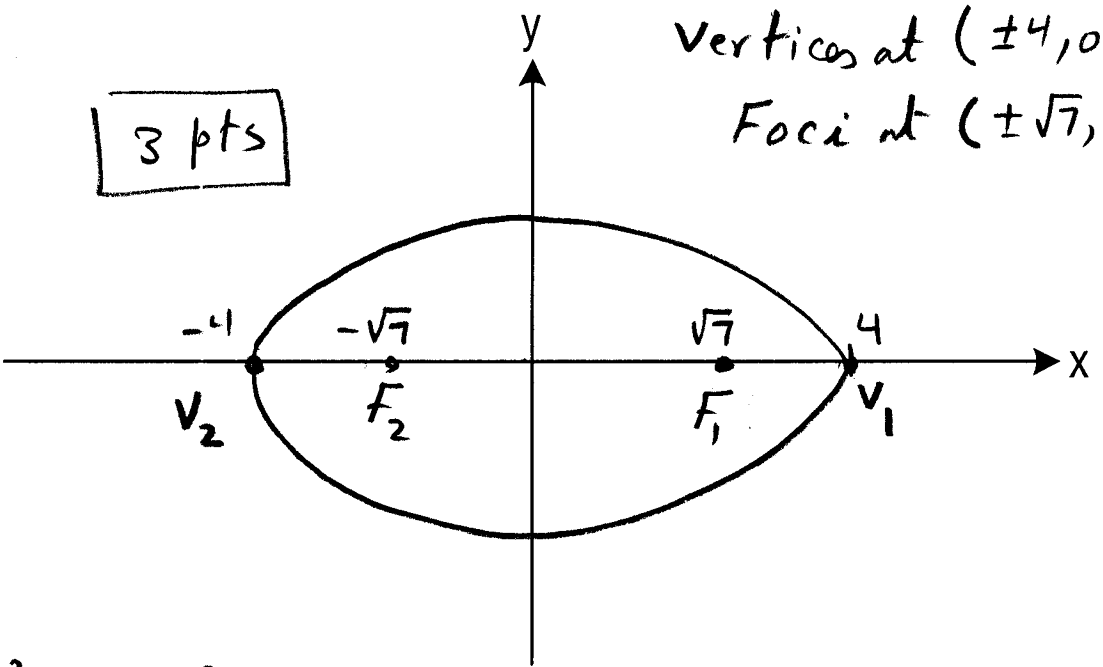
$\|w\| = \sqrt{4+64} = \sqrt{68} = 2\sqrt{17}$ ... 1 pt

$\Rightarrow \text{Proj}_w v = \frac{28}{2\sqrt{17}} = \frac{14}{\sqrt{17}}$ ... 1 pt.

9. (4-points) Sketch the graph of the ellipse $9x^2 + 16y^2 = 144$ showing its vertices and foci.

Vertices at $(\pm 4, 0)$
Foci at $(\pm\sqrt{7}, 0)$

3 pts



$\frac{x^2}{16} + \frac{y^2}{9} = 1 \Rightarrow a = 4, \ b = 3$, and $c = \sqrt{7}$

--- 1 pt

10. (5-points) Find the equation in the standard form of the parabola that has vertex $(3,-5)$, has its axis of symmetry parallel to the x-axis, and passes through the point $(4,3)$.

$$(y-k)^2 = 4p(x-h) \quad \text{--- 1 pt}$$

$$\Rightarrow \quad (y+5)^2 = 4p(x-3) \quad \text{--- 1 pt}$$

$(4,3)$ on the graph $\Rightarrow$

$$8^2 = 4p(1) \quad \text{--- 1 pt}$$

$$p = 16 \quad \text{--- 1 pt}$$

$\Rightarrow$ The required equation is

$$(y+5)^2 = 64(x-3) \quad \text{--- 1 pt}$$

11. (4-points) Solve $\cos x - \cos^2 \frac{x}{2} = -\frac{3}{4}$, $0 \le x < 2\pi$.

$$\cos x = 2\cos^2 \frac{x}{2} - 1 \Rightarrow \cos^2 \frac{x}{2} = \frac{1}{2}\cos x + \frac{1}{2} \quad \text{--- 1 pt}$$

$$\Rightarrow \cos x - \cos^2 \frac{x}{2} = \cos x - \frac{1}{2}\cos x - \frac{1}{2} = -\frac{3}{4}$$

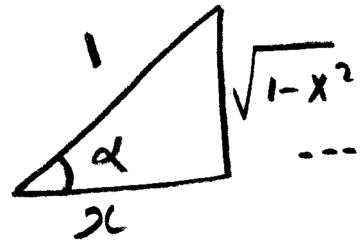
$$\Rightarrow \frac{1}{2}\cos x = -\frac{1}{4} \Rightarrow \cos x = -\frac{1}{2} \quad \text{--- 1 pt}$$

$$\Rightarrow x = \frac{2\pi}{3} \text{ or } x = \frac{4\pi}{3} \quad \text{--- 2 pts.}$$

12. (4-points) Verify the identity $\tan(2\cos^{-1} x) = \dfrac{2x\sqrt{1-x^2}}{2x^2-1}$.

Let $\alpha = \cos^{-1} x \implies x = \cos\alpha$ ... 1 pt

$\tan^{-1}(2\cos^{-1} x) = \tan 2\alpha$



... 1 pt

$$= \frac{2\tan\alpha}{1-\tan^2\alpha} \quad \text{... 1 pt}$$

$$= \frac{\dfrac{2\sqrt{1-x^2}}{x}}{1-\dfrac{1-x^2}{x^2}} = \frac{2x\sqrt{1-x^2}}{2x^2-1} \quad \text{... 1 pt}$$

13. (4-points) Find the domain and range of the function

$$f(x) = \pi + \sin^{-1}(x-1) .$$

Write your answers in interval notation.

Domain: $-1 \leq x-1 \leq 1$ ... 1 pt

$\implies 0 \leq x \leq 2$

$\implies$ Domain $= [0, 2]$ ... 1 pt

Range: $-\pi/2 \leq \sin^{-1}(x-1) \leq \frac{\pi}{2}$ ... 1 pt

$\implies \pi/2 \leq \pi + \sin^{-1}(x-1) \leq \frac{3\pi}{2}$

$\implies$ Range $= \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ ... 1 pt