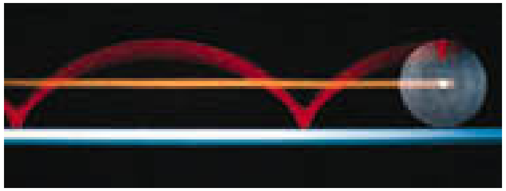
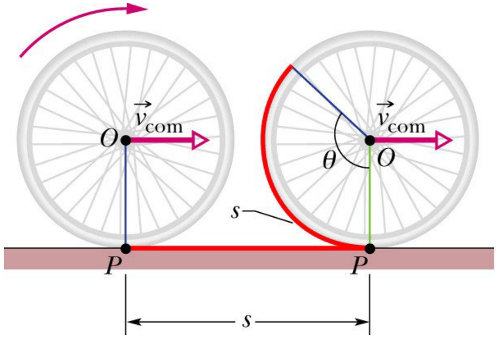
1. Rolling



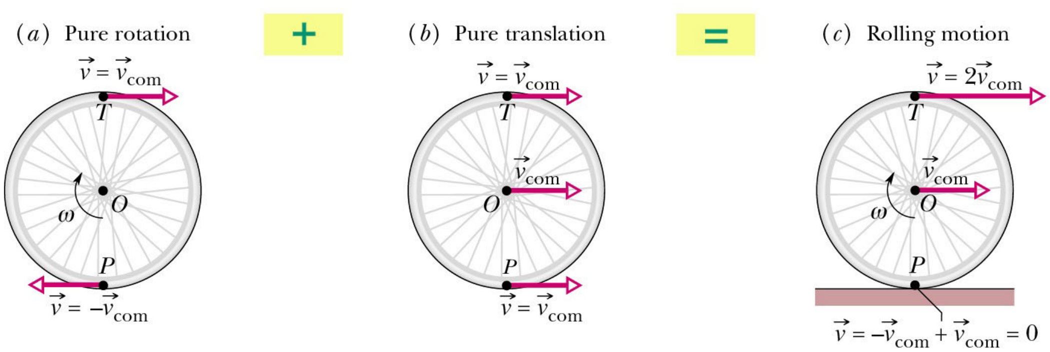
The center of the disc moves forward in pure translational motion. A point on the rim of the disc traces out a curve called the cycloid.



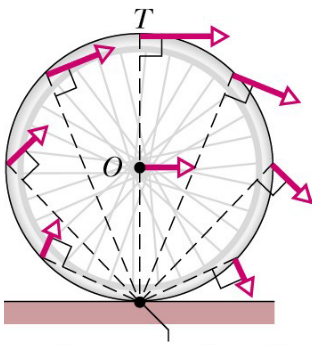
The center of mass O of the wheel moves forward at constant speed v_{com}. The point P on the street where the wheel makes contact also moves forward at speed v_{com}, so that it always remains directly below O.

$s = \theta R$ differentiating with respect to time $\boxed{v_{com} = \omega R}$

Rolling as Rotation + Translation



Rolling as pure Rotation



Rolling motion of a wheel may be viewed as pure rotation about an axis where the wheel contacts the street as the wheel moves, i.e. the point P. The angular speed about the rotation axis P is also ω.

Proof: Calculate the linear speed at the top of the rolling wheel assuming the angular speed to be ω.

$$v_{top} = (\omega)(2R) = 2(\omega R) = 2v_{com}$$

The same result as we had before, thus proving the assumption!

2. The Kinetic Energy of Rolling

Consider rolling as pure rotation about the axis P. Then, $K = \frac{1}{2} I_p \omega^2$

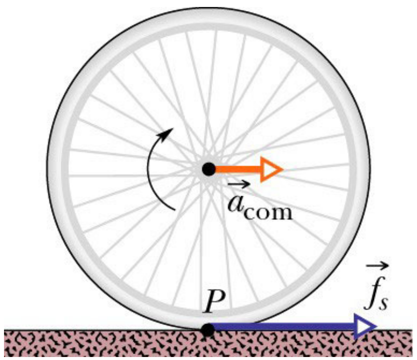
$$I_p = I_{com} + MR^2 \qquad K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} MR^2\omega^2$$

$$\boxed{K = \frac{1}{2} I_{com}\omega^2 + \frac{1}{2} Mv_{com}^2}$$

A rolling object has two types of kinetic energy: a rotational kinetic energy ($\frac{1}{2} I_{com}\omega^2$) due to its rotation about its center of mass and a translational kinetic energy ($\frac{1}{2} M v_{com}^2$) due to translation of its center of mass.

3. The forces of Rolling

Friction and Rolling



If the wheel rolls *without* sliding (*smooth rolling*) and is accelerating, then from $v_{com} = \omega R$,

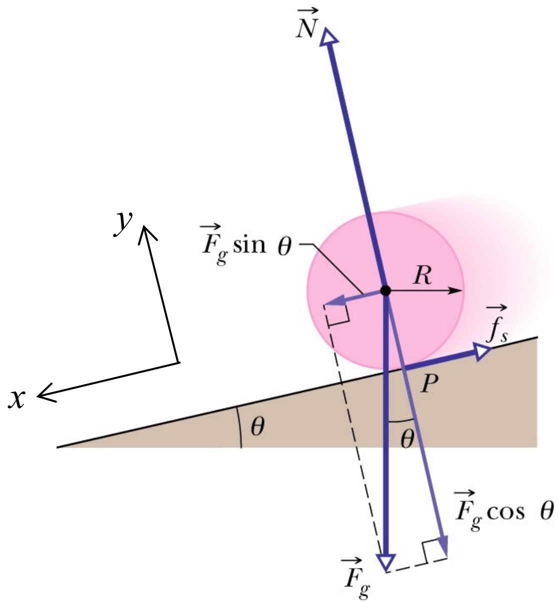
$$\frac{d v_{com}}{d t} = a_{com} = \frac{R d\omega}{d t}$$

$$\boxed{a_{com} = \alpha R} \quad \textit{(smooth rolling)}$$

where $\vec{a}_{com}$ is the linear acceleration of the center of mass and α is the angular acceleration.

- The force to provide for ma_{com} is the static frictional force (assuming the wheel rolls without sliding).
- Therefore, for a wheel to roll without sliding, the maximum static frictional force, $\mu_s N$ between the wheel and the ground must be greater than ma_{com}.
- $\vec{f}_s$ and $\vec{a}_{com}$ point to the right if the wheel if the wheel rotates faster, for example, at the start of a bicycle race.
- Do not assume that $\vec{f}_s$ is equal to the maximum value of $\mu_s N$

Rolling down a Ramp



$$\Sigma\vec{F} = M\vec{a} \;\Rightarrow\; Mg\sin\theta - f_s = M\,a_{\text{com}} \qquad (1)$$

The positive direction here is chosen to be down the plane.

Do not assume that $\vec{f}_s$ is at its maximum value of $\mu_s N$. The value of $\vec{f}_s$ self-adjusts so the body rolls without sliding.

$$\Sigma\vec{\tau} = I\vec{\alpha} \;\Rightarrow\; R\,f_s = I_{com}\,\alpha \qquad (2)$$

α is counterclockwise and positive.

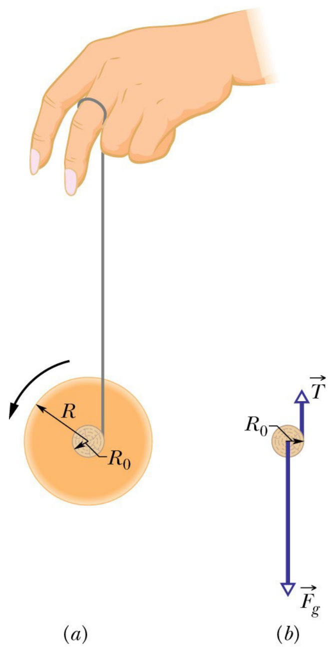
$$a_{com} = \alpha R$$

where a_{com} points down plane

Therefore from (2) → $f_s = I_{com}\dfrac{a_{com}}{R^2}$

and substituting this in (1) → $$a_{com} = \frac{g\sin\theta}{1 + I_{com}/M R^2}$$ *Note that a positive a_{com} points down plane.*

4. <u>Yo-Yo</u>



The yo-yo can be considered as a rolling down a ramp:

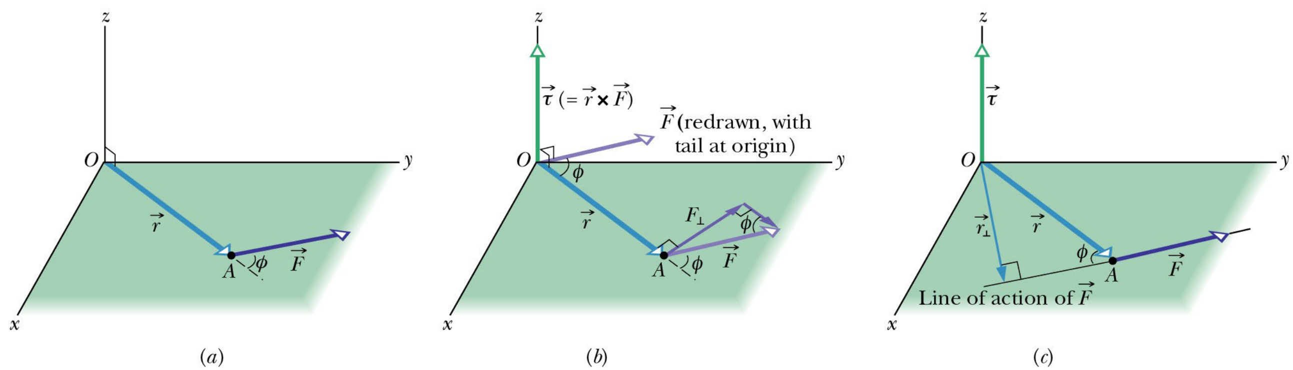
- Instead of rolling down a *ramp* at angle θ with the horizontal, the yo-yo rolls down a *string* at angle $\theta = 90°$ with the horizontal.
- Instead of rolling on its outer surface at radius R, the yo-yo rolls on an axle of radius R_o.
- Instead of being slowed by frictional force f_s, the yo-yo is slowed by the net force T on it from the string.

So we would again get the same expression for the acceleration as for rolling with $\theta = 90°$.

$$a_{com} = \frac{g}{1 + I_{com} / M R_o^2}$$

5. Torque Revisited

In chapter 11 we defined torque τ for a rigid body that can rotate about a fixed axis. Now we extend the definition of torque to apply it to an individual particle that moves along any path relative to a fixed *point* (rather than a fixed *axis*).



$$\vec{\tau} = \vec{r} \, x \, \vec{F}$$ *(torque defined)*

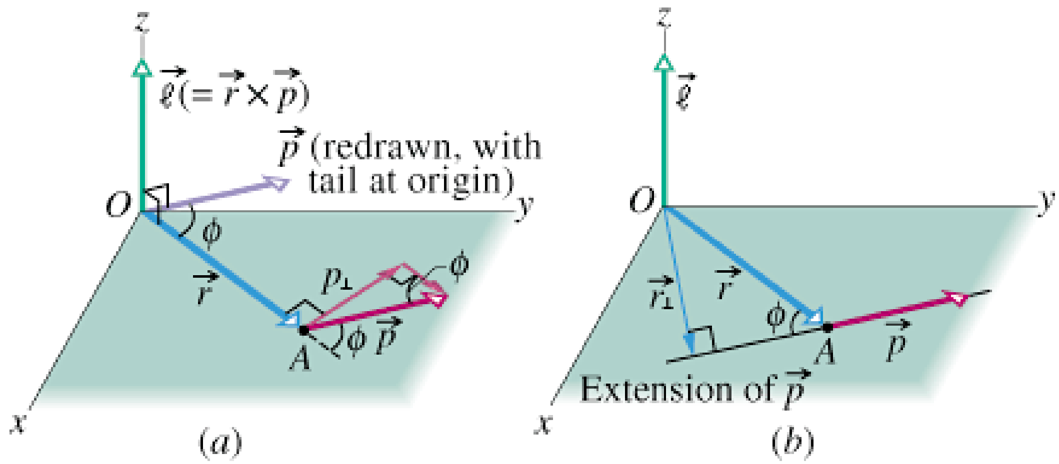
magnitude:

$$\tau = r F \sin\varphi$$

$\tau = r F_\perp$ — $F_\perp$ *is the component of F perpendicular to F*

$\tau = r_\perp F$ — $r_\perp$ is the moment arm of F (the perpendicular distance between O and the line of action of F)

6. Angular Momentum



$$l=\vec{r}\,x\,\vec{p}=m(\vec{r}\,x\,\vec{v})$$

- The *SI* unit of angular momentum is $kg\cdot m^2/s=J\cdot s$.
- Angular momentum is a "vector", the direction is determined by the right hand rule.
- The magnitude of angular momentum is

$$l=r\,m\,v\sin\varphi$$

- where φ is the angle between $\vec{r}$ and $\vec{p}$ when these two vectors are arranged tail to tail

$$magnitude:\begin{cases} l=r\,p_\perp=r\,m\,v_\perp \\ l=r_\perp p=r_\perp m v \end{cases}$$

7. Newton's Second Law in Angular Form

$$\vec{F}_{net} = \frac{d\,\vec{p}}{d\,t} \qquad \textit{(single particle)}$$

$$\Rightarrow \qquad \boxed{\vec{\tau}_{net} = \frac{d\,\vec{l}}{d\,t}} \qquad \textit{(single particle)}$$

- Note that the torque $\vec{\tau}$ and angular momentum $\vec{\alpha}$ must be defined with respect to the same origin.

- Proof: $\vec{l} = m(\vec{r}\,\mathrm{x}\,\vec{v})$

$$\frac{d\,\vec{l}}{d\,t} = m(\vec{r}\,\mathrm{x}\frac{d\,\vec{v}}{d\,t} + \frac{d\,\vec{r}}{d\,t}\mathrm{x}\,\vec{v}) = m(\vec{r}\,\mathrm{x}\,\vec{a} + \vec{v}\,\mathrm{x}\,\vec{v})$$

Because $\vec{v}\,\mathrm{x}\,\vec{v} = 0$, this leads to

$$\frac{d\,\vec{l}}{d\,t} = m(\vec{r}\,\mathrm{x}\,\vec{a}) = \vec{r}\,\mathrm{x}\,m\,\vec{a} = \vec{r}\,\mathrm{x}\,\vec{F}_{\mathrm{net}} = \sum(\vec{r}\,\mathrm{x}\,\vec{F}) = \vec{\tau}_{\mathrm{net}}$$

Therefore, $\vec{\tau}_{net} = \frac{d\,\vec{l}}{d\,t}$

8. Angular Momentum of a System of Particles

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \ldots + \vec{l}_n = \sum_{i=1}^{n} \vec{l}_i \qquad \textit{(L = total angular momentum)}$$

$$\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{l}_i}{dt} = \sum_{i=1}^{n} \vec{\tau}_{\text{net},i}$$

- $\vec{\tau}_{\text{net},i}$ is the net torque on the i^{th} particle. $\sum_{i=1}^{n} \vec{\tau}_{\text{net},i}$ is the sum of all the torque (internal and and external) on the system. However the internal torques sums to zero. Let $\vec{\tau}_{net}$ represent the net external torque on the system.

$$\Rightarrow \qquad \boxed{\vec{\tau}_{net} = \frac{d\vec{L}}{dt}} \qquad \textit{(system of particles)}$$

- The *net external torque* $\vec{\tau}_{\text{net}}$ acting on a system of particles is equal to the *time rate of change* of the system's total *angular momentum* $\vec{L}$.

9. Angular Momentum of a rigid Body Rotating About a Fixed Axis

$$L = I\omega \qquad \textit{(rigid body, fixed axis)}$$

- L is the angular momentum of the rigid body about the rotation axis and I is the moment of inertia of the rigid body about the same axis.

TABLE 12-1 More Corresponding Variables and Relations for Translational and Rotational Motion[a]

Translational		Rotational	
Force	$\vec{F}$	Torque	$\vec{\tau}\,(=\vec{r}\times\vec{F})$
Linear momentum	$\vec{p}$	Angular momentum	$\vec{l}\,(=\vec{r}\times\vec{p})$
Linear momentum[b]	$\vec{P}\,(=\sum\vec{p}_i)$	Angular momentum[b]	$\vec{L}\,(=\sum\vec{l}_i)$
Linear momentum[b]	$\vec{P}=M\,\vec{v}_{\mathrm{com}}$	Angular momentum[c]	$L=I\,\omega$
Newton's second law[b]	$\vec{F}_{\mathrm{net}}=\dfrac{d\,\vec{P}}{d\,t}$	Newton's second law[b]	$\vec{\tau}_{\mathrm{net}}=\dfrac{d\,\vec{L}}{d\,t}$
Conservation law[d]	$\vec{P}=\text{a constant}$	Conservation law[d]	$\vec{L}=\text{a constant}$

[a] See also Table 11-3.
[b] For systems of particles, including rigid bodies.
[c] For a rigid body about a fixed axis, with L being the component along that axis.
[d] For a closed, isolated system.

10. <u>Conservation of Angular Momentum</u>

- If no external net torque acts on the system, then:

$$\vec{L} = \text{a constant} \quad \text{(isolated system)}$$

- This result is called the *Law of Conservation of Angular Momentum*

$$\boxed{\begin{pmatrix}\text{net angular momentum}\\ \text{at some initial time } t_i\end{pmatrix} = \begin{pmatrix}\text{net angular momentum}\\ \text{at some later time } t_f\end{pmatrix}}$$

$$\boxed{\vec{L}_i = \vec{L}_f} \qquad (\textit{ isolated system })$$

- If the *net external torque* acting on a system is *zero*, the angular momentum $\vec{L}$ of the system remains constant, no matter what changes take place within the system.

- If the component of the net *external* torque on a system *along a certain axis* is *zero*, then the *component* of the *angular momentum* of the system *along that axis* cannot change, no matter what changes take place within the system.

$$\boxed{I_i \omega_i = I_f \omega_f}$$