1. A Special Point

- ***The center of mass (com) of a body or a system of bodies is the point that moves as though all of the mass were concentrated there and all external forces were applied there.***

Click on the link to view a simulation of the trajectory of a center of mass

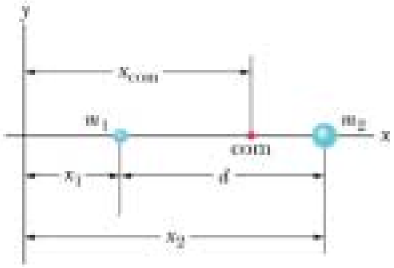
2. The Center of Mass

Systems of particles

- Consider two particles of masses m_1 and m_2 and at positions x_1 and x_2

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{M}$$

where $\boldsymbol{M = m_1 + m_2}$ is the total mass



In general for more than just two particles:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \cdots + m_n x_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

If the particles are distributed in three dimensions:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \quad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \quad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

- In the language of the vectors:

$$\vec{r} = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}} \qquad \vec{r}_{com} = x_{com}\hat{\mathbf{i}} + y_{com}\hat{\mathbf{j}} + z_{com}\hat{\mathbf{k}}$$

The three components equation can now be replaced by a single vector equation:

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

<u>Solid Bodies</u>

- A solid body can be thought as consisting of infinite number of small mass element *dm* . The particles then becomes the differential mass elements *dm* at positions (x, y, z); the sum becomes the integral:

$$x_{com} = \frac{1}{M}\int x\,dm, \quad y_{com} = \frac{1}{M}\int y\,dm, \quad z_{com} = \frac{1}{M}\int z\,dm$$

- For objects having uniform density ρ, we have both $\rho V = M$ and $\rho\, dV = dm$ where V is the volume of the object. Substituting these facts in the above equation we get:

$$x_{com} = \frac{1}{V}\int x\,dV, \quad y_{com} = \frac{1}{V}\int y\,dV, \quad z_{com} = \frac{1}{V}\int z\,dV$$

(for uniform solid objects)

3. <u>Newton's Second Law for a system of Particles</u>

$$\vec{F}_{net} = M\,\vec{a}_{com} \quad (system\ of\ \ particles)$$

- $\vec{F}_{net}$ is the net force of *all external forces* that act on the system. Forces on one part of the system from another (*internal forces*) are not included

- M is the *total mass* of the system. We assume that no mass enters or leaves the system as it moves, so that *M* remains constant. The system is said to be **closed**

- $\vec{a}_{com}$ is the acceleration of the *center of mass* of the system. This equation gives no information about the acceleration of any other point of the system.

The above vector equation is equivalent to three component equations as follows:

$$F_{net,x} = M\,a_{com,x} \qquad F_{net,y} = M\,a_{com,y} \qquad F_{net,z} = M\,a_{com,z}$$

<u>Velocity of the center of mass</u> $\vec{v}_{com}$:

Differentiating the definition of center of mass with respect to time:

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i\,\vec{r}_i \Rightarrow \quad \vec{v}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i\,\vec{v}_i$$

Differentiating this once more with respect to time results in the Newton's 2^{nd} law for system of particles as stated at the beginning

4. Linear Momentum

$$\vec{p} = m\vec{v}$$ (linear momentum of a particle)

- Linear momentum is a vector and the SI unit is kg-m / s.
- Newton actually expressed his 2nd law of motion in terms of momentum:

$$\vec{F}_{net} = \frac{d\,\vec{p}}{d\,t}$$

The time rate of change of the momentum of a particle is equal to the net force acting on the particle and is in the direction of that force.

- For constant mass m this reduces to the familiar expression as follows:

$$\vec{F}_{net} = \frac{d\,\vec{p}}{d\,t} = \frac{d}{d\,t}(m\vec{v}) = m\frac{d\,\vec{v}}{d\,t} = m\vec{a}$$

5. The Linear Momentum of a System of Particles

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_n$$

$$= \underbrace{m_1\,\vec{v}_1 + m_2\,\vec{v}_2 + m_3\,\vec{v}_3 + \cdots + m_n\,\vec{v}_n}$$

$$\downarrow$$

$$\vec{P} = M\;\vec{v}_{com}$$

This follows from the expression for $\mathbf{v}_{com}$ from the previous slide

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

Taking the time derivative of total linear momentum $\vec{P}$ of the system of particles:

$$\frac{d\,\vec{P}}{d\,t} = M\,\frac{d\,\vec{v}_{com}}{dt} = M\;\vec{a}_{com}$$

$$\Rightarrow \vec{F}_{net} = \frac{d\,\vec{P}}{d\,t}$$

For system of particles

6. <u>Conservation of Linear Momentum</u>

$$\boxed{\vec{P} = constant} \qquad \text{(closed, isolated system)}$$

- If no net external force acts on a system of particles $\vec{F}_{net} = 0$, then the total linear momentum of the system cannot change, i.e. $d\vec{P}/dt = 0$.

- This result is called the law of conservation of linear momentum. It can also be written as:

$$\boxed{\vec{P}_i = \vec{P}_f} \qquad \text{(closed, isolated system)}$$

- Depending on the forces acting on the system, linear momentum can be conserved in one or two directions but not in all directions:

> ***If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.***