Rotation with Constant Angular Acceleration :

When the angular acceleration α is constant we can derive simple expressions that give us the angular velocity ω and the angular position θ as a function of time. We could derive these equations in the same way we did in Chapter 2. Instead we will simply write the solutions by exploiting the analogy between translational and rotational motion using the following correspondence between the two motions.

Translational Motion		Rotational Motion	
x	$\leftrightarrow$	θ	
v	$\leftrightarrow$	ω	
a	$\leftrightarrow$	α	
$v = v_0 + at$	$\leftrightarrow$	$\omega = \omega_0 + \alpha t$	(eq. 1)
$x = x_o + v_0 t + \frac{at^2}{2}$	$\leftrightarrow$	$\theta = \theta + \omega_0 t + \frac{\alpha t^2}{2}$	(eq. 2)
$v^2 - v_0^2 = 2a\left(x - x_o\right)$	$\leftrightarrow$	$\omega^2 - \omega_0^2 = 2\alpha\left(\theta - \theta_0\right)$	(eq. 3)

A wheel rotates with a constant angular acceleration of $3.50\ \text{rad/s}^2$.

(A) If the angular speed of the wheel is 2.00 rad/s at $t_i = 0$, through what angular displacement does the wheel rotate in 2.00 s?

(B) Through how many revolutions has the wheel turned during this time interval?

(C) What is the angular speed of the wheel at $t = 2.00$ s?

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$$\Delta\theta = \theta_f - \theta_i = \omega_i t + \tfrac{1}{2}\alpha t^2$$
$$= (2.00\ \text{rad/s})(2.00\ \text{s}) + \tfrac{1}{2}(3.50\ \text{rad/s}^2)(2.00\ \text{s})^2$$
$$= \boxed{11.0\ \text{rad}} = (11.0\ \text{rad})(57.3°/\text{rad}) = \boxed{630°}$$

(B) Through how many revolutions has the wheel turned during this time interval?

$$\Delta\theta = 630°\left(\frac{1\ \text{rev}}{360°}\right) = \boxed{1.75\ \text{rev}}$$

(C) What is the angular speed of the wheel at $t = 2.00\ \text{s}$?

$$\omega_f = \omega_i + \alpha t = 2.00\ \text{rad/s} + (3.50\ \text{rad/s}^2)(2.00\ \text{s})$$
$$= \boxed{9.00\ \text{rad/s}}$$

A rotating wheel requires 3.00 s to rotate through 37.0 revolutions. Its angular speed at the end of the 3.00-s interval is 98.0 rad/s. What is the constant angular acceleration of the wheel?

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$\theta_f - \theta_i = \omega_i t + \frac{1}{2}\alpha t^2$ and $\omega_f = \omega_i + \alpha t$ are two equations in two unknowns ω_i and α

$\omega_i = \omega_f - \alpha t$:

$$\theta_f - \theta_i = (\omega_f - \alpha t)t + \frac{1}{2}\alpha t^2 = \omega_f t - \frac{1}{2}\alpha t^2$$

$$37.0 \text{ rev}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 98.0 \text{ rad/s}(3.00 \text{ s}) - \frac{1}{2}\alpha(3.00 \text{ s})^2$$

$232 \text{ rad} = 294 \text{ rad} - (4.50 \text{ s}^2)\alpha$:

$$\alpha = \frac{61.5 \text{ rad}}{4.50 \text{ s}^2} = \boxed{13.7 \text{ rad/s}^2}$$

Example: An electric motor rotating a grinding wheel at 100 rev/min is switched off. Assuming constant negative angular acceleration of magnitude 2.00 rad/s2,

(a) How long does it take the wheel to stop?

(b) Through how many radians does it turn during the time found in (a)?

Example: An electric motor rotating a grinding wheel at 100 rev/min is switched off. Assuming constant negative angular acceleration of magnitude 2.00 rad/s2,

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(b) Through how many radians does it turn during the time found in (a)?

$$100\,\frac{\text{rev}}{\text{min}} = \left(100\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{2\pi\,\text{rad}}{1\,\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{s}}\right) = 10.5\,\frac{\text{rad}}{\text{s}}$$

When the wheel has stopped then of course its angular velocity is zero

$$\omega = \omega_0 + \alpha t \qquad \Longrightarrow \qquad t = \frac{(\omega - \omega_0)}{\alpha}$$

$$t = \frac{(0 - 10.5\,\frac{\text{rad}}{\text{s}})}{(-2.00\,\frac{\text{rad}}{\text{s}^2})} = 5.24\,\text{s}$$

$$\theta = \tfrac{1}{2}(\omega_0 + \omega)t = \tfrac{1}{2}(10.5\,\tfrac{\text{rad}}{\text{s}} + 0)(5.24\,\text{s}) = 27.5\,\text{rad}\ .$$

A disk 8.00 cm in radius rotates at a constant rate of 1 200 rev/min about its central axis. Determine (a) its angular speed, (b) the tangential speed at a point 3.00 cm from its center, (c) the radial acceleration of a point on the rim, and (d) the total distance a point on the rim moves in 2.00 s.

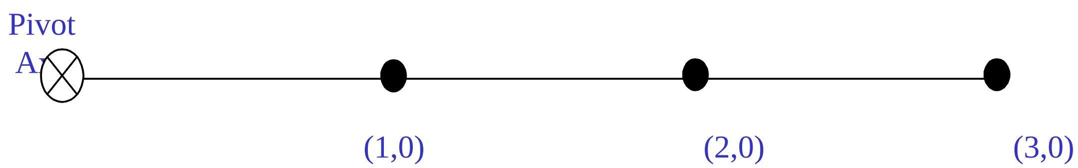
(a) $\omega = 2\pi f = \dfrac{2\pi \text{ rad}}{1 \text{ rev}}\left(\dfrac{1\,200 \text{ rev}}{60.0 \text{ s}}\right) = \boxed{126 \text{ rad/s}}$

(b) $v = \omega r = (126 \text{ rad/s})(3.00 \times 10^{-2} \text{ m}) = \boxed{3.77 \text{ m/s}}$

(c) $a_c = \omega^2 r = (126)^2(8.00 \times 10^{-2}) = 1\,260 \text{ m/s}^2$ so $\mathbf{a}_r = \boxed{1.26 \text{ km/s}^2 \text{ toward the center}}$

(d) $s = r\theta = \omega r t = (126 \text{ rad/s})(8.00 \times 10^{-2} \text{ m})(2.00 \text{ s}) = \boxed{20.1 \text{ m}}$

Example: As shown in the figure, three masses, of 1.5 kg each, are fastened at fixed position to a very light rod pivoted at one end. Find the moment of inertia for the rotation axes shown.



$$I = \sum_i^3 m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$= m(r_1^2 + r_2^2 + r_3^2) = 1.5(1^2 + 2^2 + 3^2) = 21 \text{ kg.m}^2$$

Example: Calculate the rotational inertia of a wheel that has a kinetic energy of 24, 400 J when rotating at 602 rev/min.

Find the angular speed ω of the wheel, in $\frac{\text{rad}}{\text{s}}$:

$$\omega = 602\,\frac{\text{rev}}{\text{min}}\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right) = 63.0\,\frac{\text{rad}}{\text{s}}$$

We have ω and the kinetic energy of rotation, K_{rot}, so we can find the rotational kinetic energy from

$$K_{\text{rot}} = \tfrac{1}{2}I\omega^2 \qquad \Longrightarrow \qquad I = \frac{2K_{\text{rot}}}{\omega^2}$$

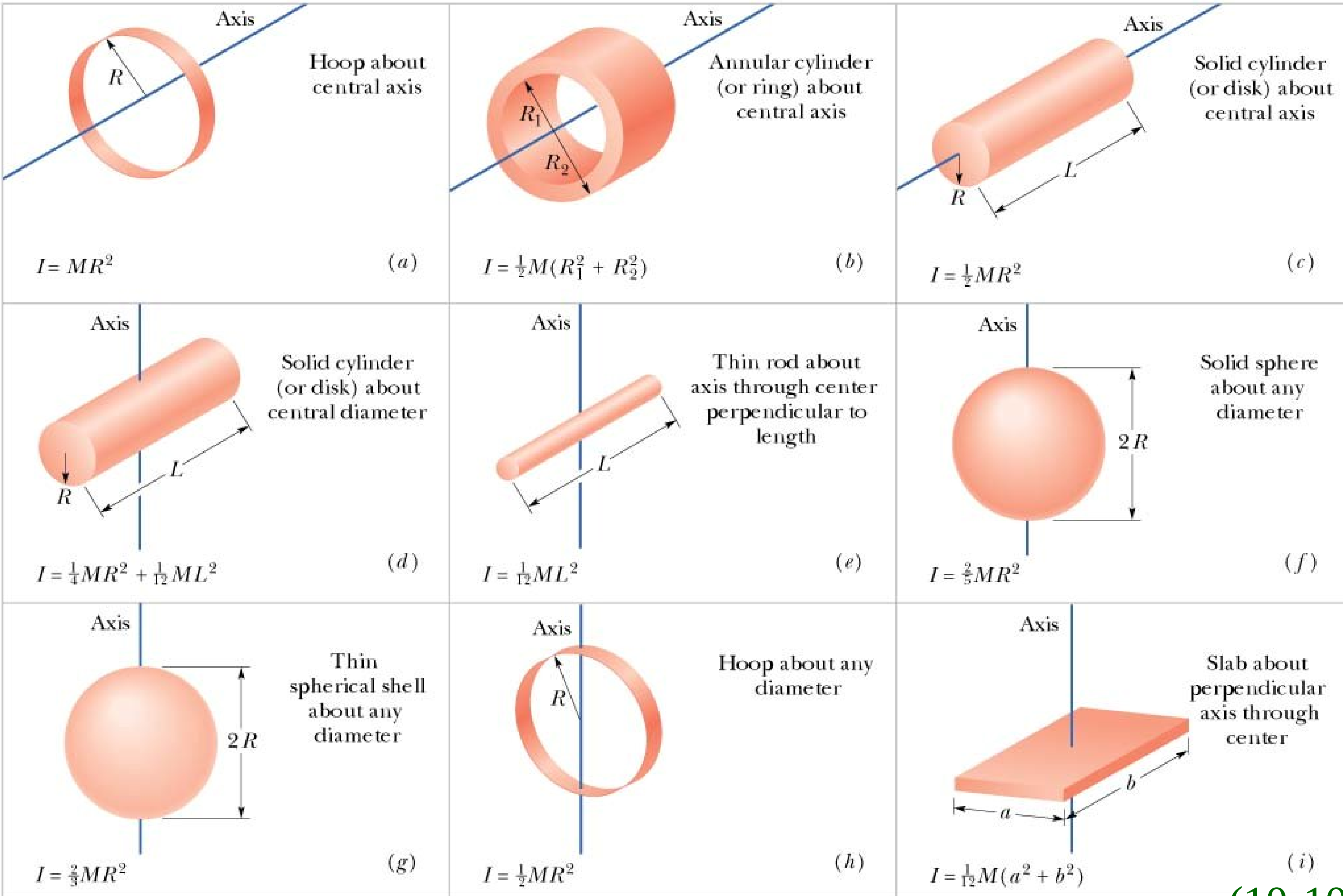
We get:

$$I = \frac{2(24,400\ \text{J})}{(63.0\,\frac{\text{rad}}{\text{s}})^2} = 12.3\,\text{kg}\cdot\text{m}^2$$

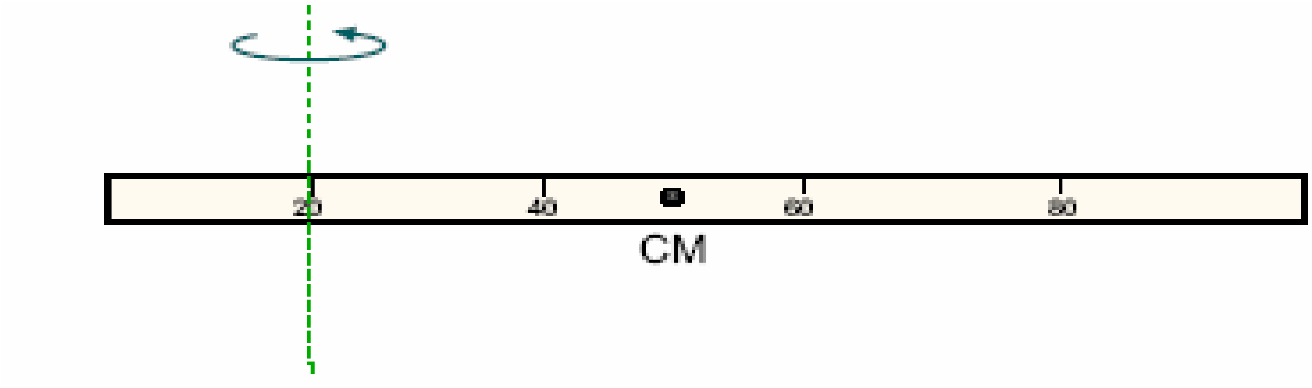
The moment of inertia of the wheel is $12.3\,\text{kg}\cdot\text{m}^2$.

In the table below we list the rotational inertias for some rigid bodies.

$$I = \int r^2 dm$$



Example: Calculate the rotational inertia of a meter stick with mass 0.56 kg, about an axis perpendicular to the stick and located at the 20 cm mark.



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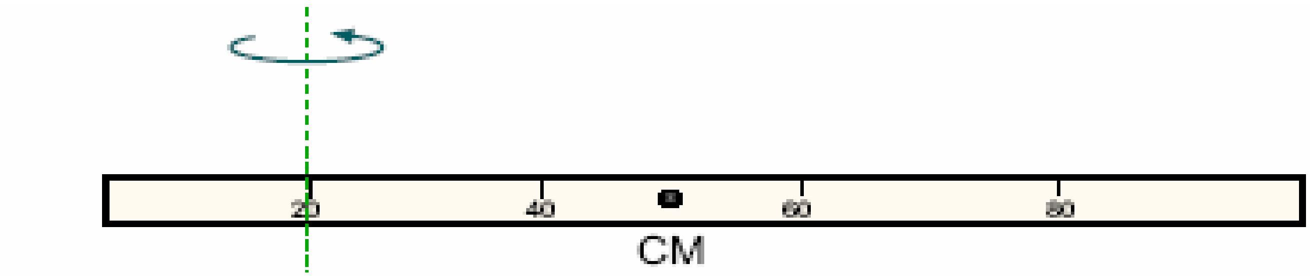


$$I_{CM} = \frac{1}{12}ML^2 = \frac{1}{12}(0.56\,\mathrm{kg})(1.00\,\mathrm{m})^2 = 4.7 \times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2$$

The rotational inertia about *our* axis will not be the same.

We note that our axis is displaced from the one through the CM by 30 cm. Then the Parallel Axis Theorem (Eq. 1.16) tells us that the moment of inertia about our axis is given by

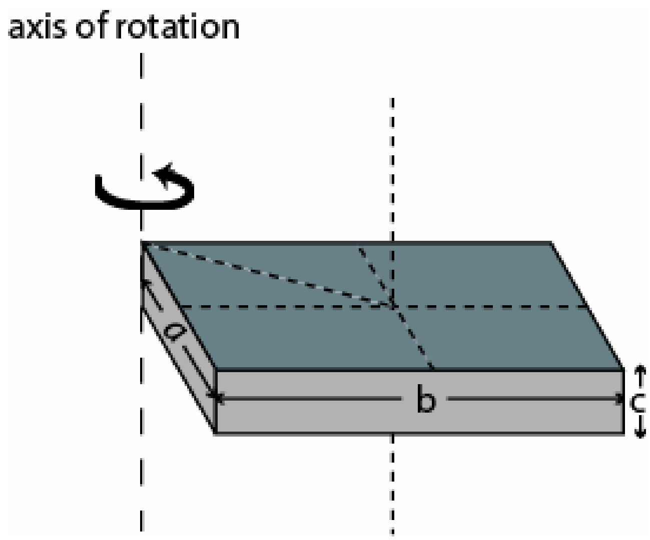
$$I = I_{CM} + MD^2$$

where $I_{CM} = 4.7 \times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2$, as we've already found, M is the mass of the rod and D is the distance the axis is displaced (parallel to itself), namely 30 cm. We get:

$$I = 4.7 \times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2 + (0.56\,\mathrm{kg})(0.30\,\mathrm{m})^2 = 9.7 \times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2$$

So the rotational inertia of the stick *about the given axis* is $0.097\,\mathrm{kg}\cdot\mathrm{m}^2$.

A uniform **slab** of dimensions: *a* = 60 cm, *b* = 80 cm, and *c* = 2.0 cm (see Fig. 6) has a mass of 6.0 kg. Its rotational inertia about an axis perpendicular to the larger face and passing through one corner of the slab is: (Ans: 2.0 kg.m2)



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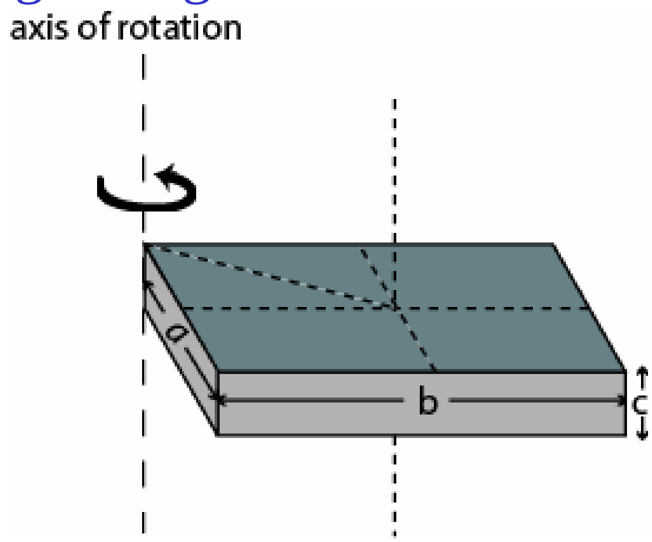
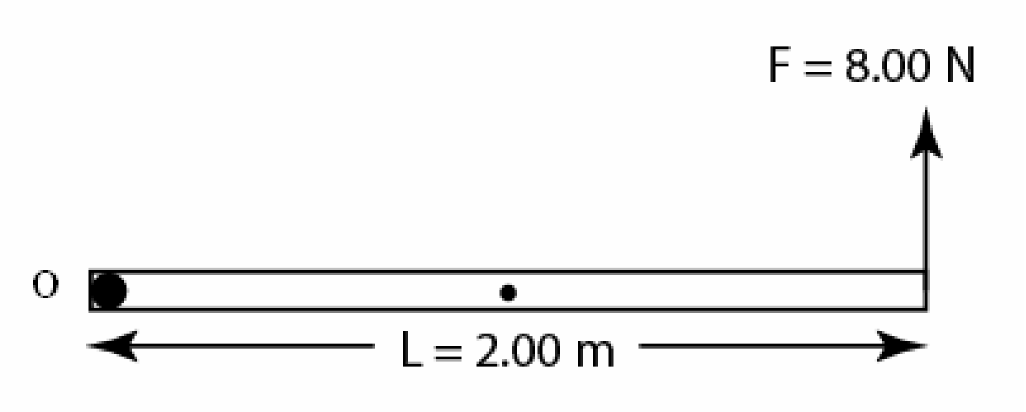


Use the equation: $I = I_o + MD^2 \quad \Rightarrow \quad I = \frac{M}{12}(a^2 + b^2) + M\left[(\frac{a}{2})^2 + (\frac{b}{2})^2\right] = \frac{M}{3}\underbrace{(a^2 + b^2)}_{1}$

$$\Rightarrow I = \underline{2.0\ \text{kg.m}^2}$$

A uniform thin rod of mass M = 3.00 kg and length L = 2.00 m is pivoted at one end O and acted upon by a force F = 8.00 N at the other end as shown in Figure 5. The angular acceleration of the rod at the moment the rod is in the horizontal position as shown in this figure is: (Ans 4.0 rad/s2 counterclockwise)



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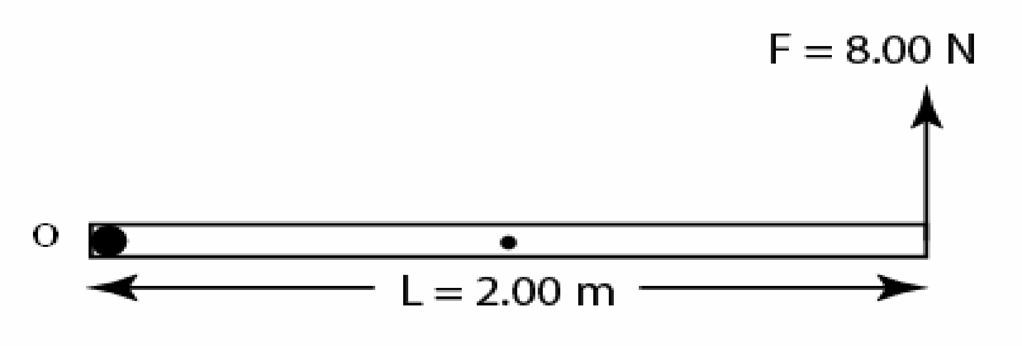
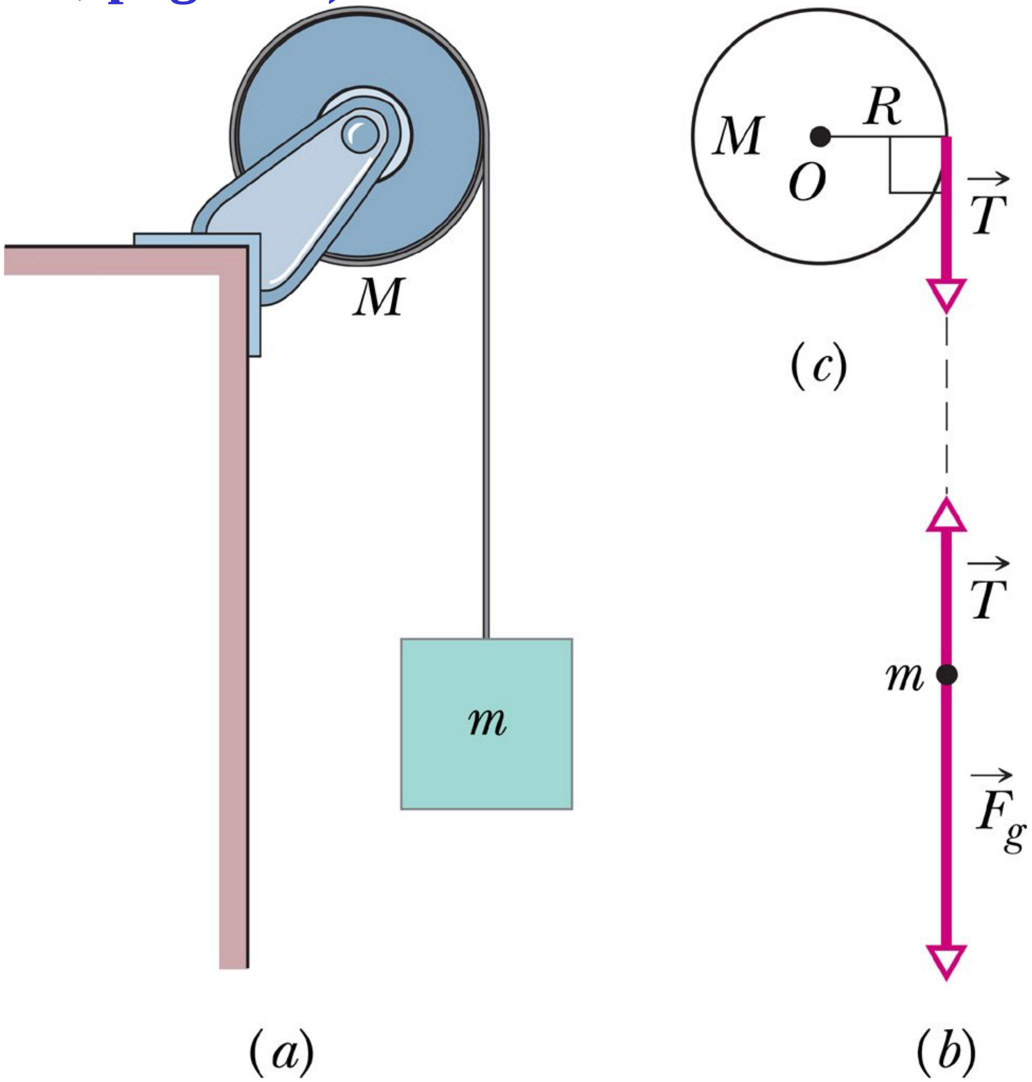


$$I_o = I_{CM} + Md^2 = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2;$$

$$\tau = I\alpha = FL \Rightarrow \alpha = \frac{FL}{I} = \frac{8\times 2}{\frac{1}{3}(3)2^2} = 4\frac{\text{rad}}{\text{s}}$$

Example: (Sample problem 10-8, page 258)

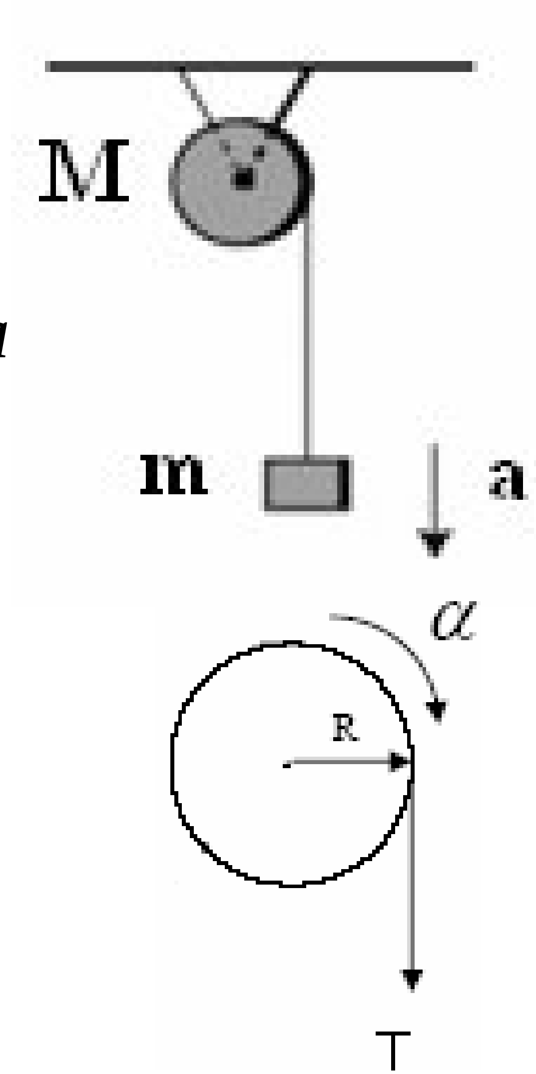
In the Fig, Shows a uniform **disk**, with mass **M=2.5 kg** and radius **R=20cm**, mounted on a fixed horizontal axle. A block with mass **m=1.2** kg hangs from a massless cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the angular acceleration of the disk, and the tension in the cord. The cord does not slip, and there is no friction at the axle.



$$ma = mg - T \quad (1),$$

$$I\alpha = TR \quad (2),$$

$$I = \frac{1}{2}MR^2, \qquad \alpha = \frac{a}{R} \qquad T = \frac{1}{2}Ma$$



$$ma = mg - \frac{1}{2}Ma$$

$$\Rightarrow a = \left(\frac{2m}{M+2m}\right)g = \frac{2\times1.2}{2.5+2\times1.2} = \underline{4.8\ \text{m/s}^2}$$

$$\Rightarrow T = \frac{1}{2}Ma = \underline{6.0\ \text{N}}$$

$$\alpha = \frac{a}{R} = \underline{24\ \text{rad/s}^2}$$

The power output of an automobile engine is advertised to be 200 hp at 6000 rpm. What is the corresponding torque?

The power output of an automobile engine is advertised to be 200 hp at 6000 rpm. What is the corresponding torque?

$$P = 200\ \text{hp} = 200\ \text{hp}\left(\frac{746\ \text{W}}{1\ \text{hp}}\right) = 1.49 \times 10^5\ \text{W}$$

$$\omega_z = 6000\ \text{rev/min} = \left(\frac{6000\ \text{rev}}{1\ \text{min}}\right)\left(\frac{2\pi\ \text{rad}}{1\ \text{rev}}\right)\left(\frac{1\ \text{min}}{60\ \text{s}}\right)$$

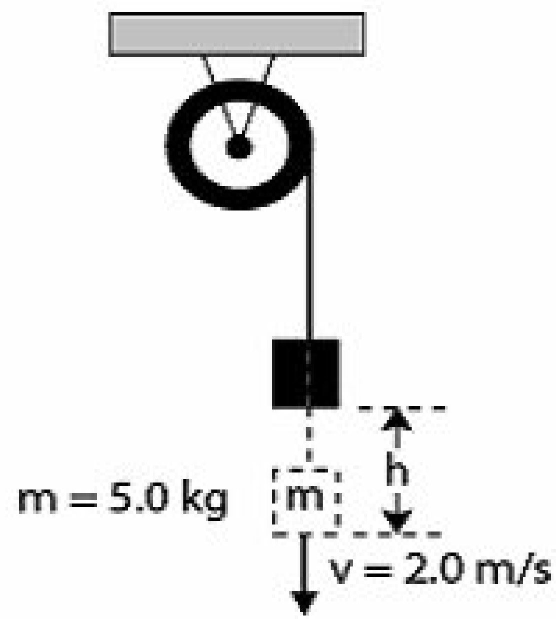
$$= 628\ \text{rad/s}$$

$$\tau_z = \frac{P}{\omega_z} = \frac{1.49 \times 10^5\ \text{N} \cdot \text{m/s}}{628\ \text{rad/s}} = 237\ \text{N} \cdot \text{m}$$

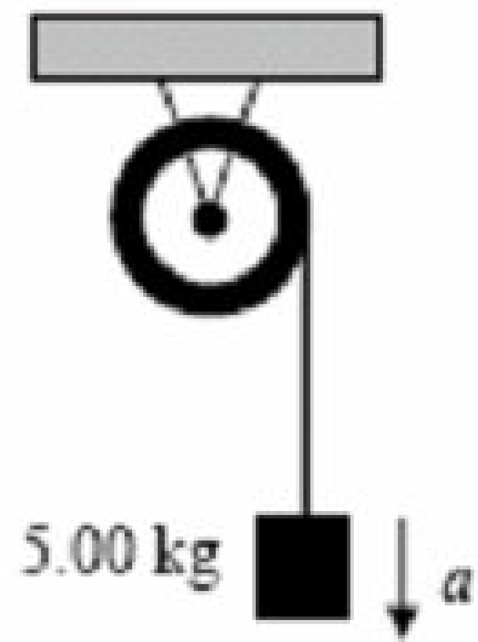
Analogies Between Translational and Rotational Motion

Translational Motion		Rotational Motion
x	$\leftrightarrow$	θ
v	$\leftrightarrow$	ω
a	$\leftrightarrow$	α
$v = v_0 + at$	$\leftrightarrow$	$\omega = \omega_0 + \alpha t$
$x = x_o + v_0 t + \frac{at^2}{2}$	$\leftrightarrow$	$\theta = \theta_0 + \omega_0 t + \frac{\alpha t^2}{2}$
$v^2 - v_0^2 = 2a\left(x - x_o\right)$	$\leftrightarrow$	$\omega^2 - \omega_0^2 = 2\alpha\left(\theta - \theta_0\right)$
$K = \frac{mv^2}{2}$	$\leftrightarrow$	$K = \frac{I\omega^2}{2}$
m	$\leftrightarrow$	I
$F = ma$	$\leftrightarrow$	$\tau = I\alpha$
F	$\leftrightarrow$	τ
$P = Fv$	$\leftrightarrow$	$P = \tau\omega$

T072 Q#16: A wheel of radius $R = 0.20$ m is mounted on a fixed frictionless horizontal axis. The rotational inertia I of the wheel about this axis is 0.50 kg.m^2. A massless cord wrapped around the circumference of the wheel is attached to a $m = 5.0$ kg box (Fig. 8). The box is then released from rest. When the box has a speed of v = 2.0 m/s, the distance (h) through which the box has fallen is:



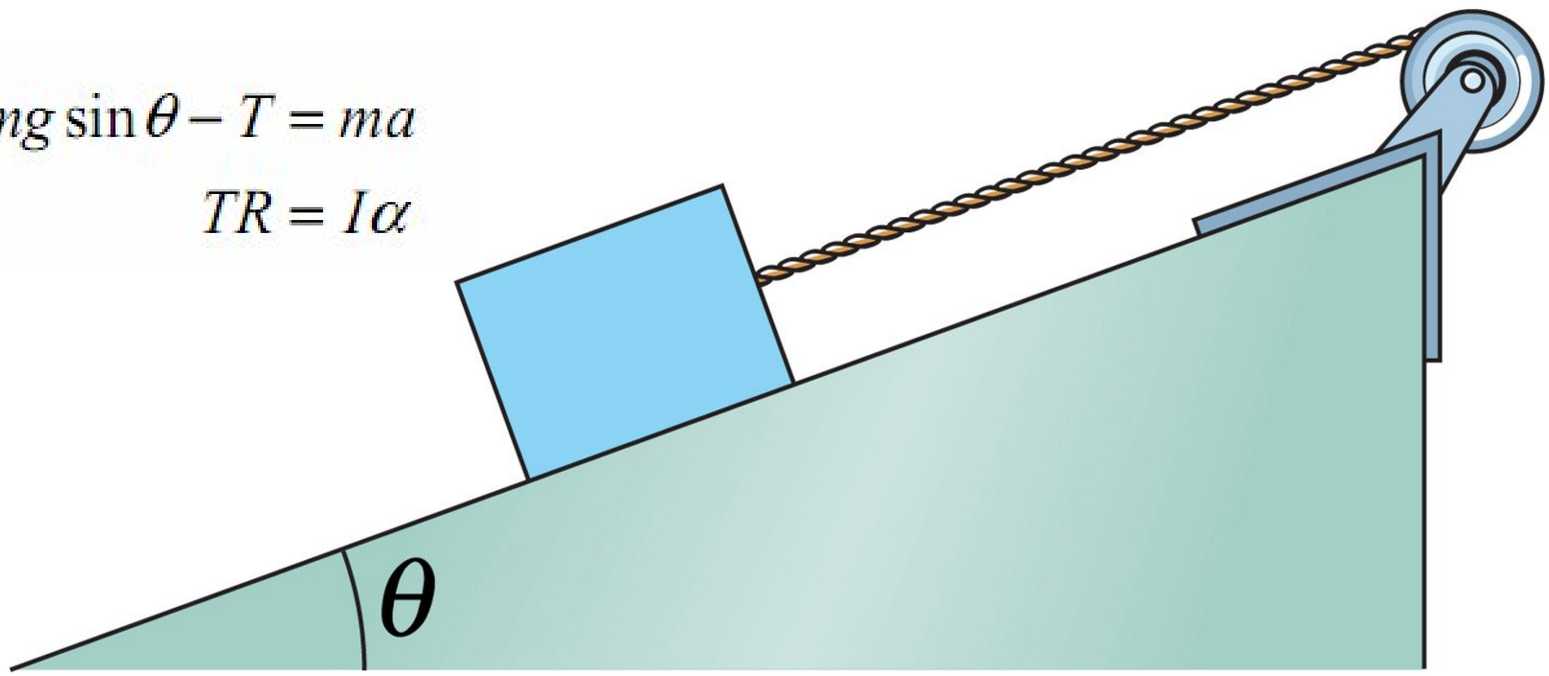
T071 Q15. A 5.00 kg block hangs from a cord which is wrapped around the rim of a frictionless pulley as shown in the figure. What is the acceleration, *a*, of the block as it moves down? (The rotational inertia of the pulley is 0.200 kg· m^2 and its radius is 0.100 m.) (Ans: 1.96 m/ s^2)



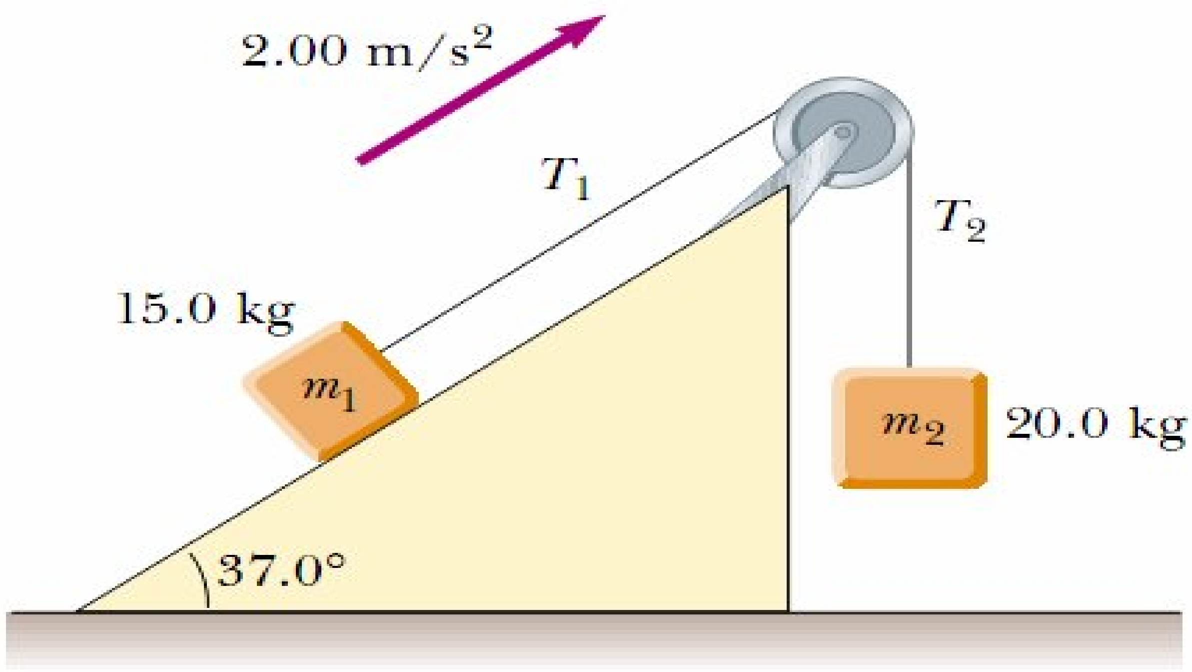
In Fig, a wheel of radius R= 0.20 m is mounted on a frictionless horizontal axle. A massless cord is wrapped around the wheel and attached to a 2.0 kg box that slides on frictionless surface inclined at angle theta =20 degree with the horizontal. The box accelerated down the surface at 2.0 m/s^2. What is the rotational inertia of the wheel about the axle?

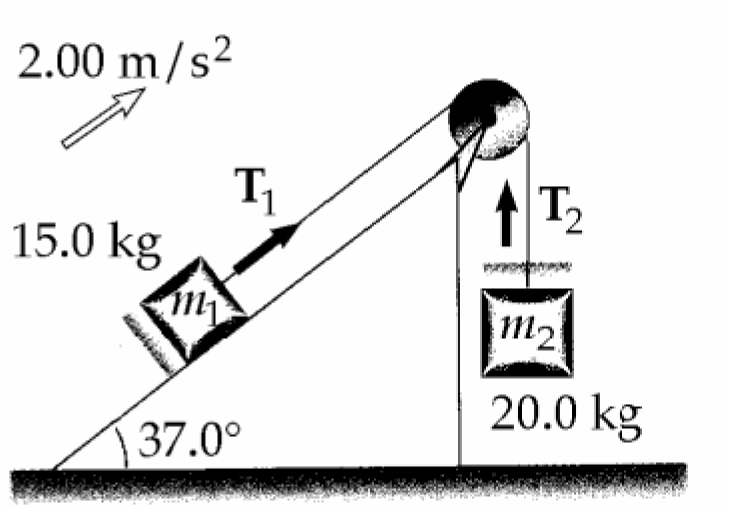
$$mg\sin\theta - T = ma$$

$$TR = I\alpha$$



Two blocks, as shown in Figure P10.71, are connected by a string of negligible mass passing over a pulley of radius 0.250 m and moment of inertia I. The block on the frictionless incline is moving up with a constant acceleration of 2.00 m/s^2. (a) Determine T_1 and T_2, the tensions in the two parts of the string. (b) Find the moment of inertia of the pulley.





(a)

$$m_2 g - T_2 = m_2 a$$

$$T_2 = m_2 (g - a) = 20.0\ \text{kg}\left(9.80\ \text{m/s}^2 - 2.00\ \text{m/s}^2\right) = \boxed{156\ \text{N}}$$

$$T_1 - m_1 g \sin 37.0° = m_1 a$$

$$T_1 = (15.0\ \text{kg})(9.80 \sin 37.0° + 2.00)\ \text{m/s}^2 = \boxed{118\ \text{N}}$$

(b)

$$(T_2 - T_1)R = I\alpha = I\left(\frac{a}{R}\right)$$

$$I = \frac{(T_2 - T_1)R^2}{a} = \frac{(156\ \text{N} - 118\ \text{N})(0.250\ \text{m})^2}{2.00\ \text{m/s}^2} = \boxed{1.17\ \text{kg}\cdot\text{m}^2}$$

Find the net torque on the wheel in Figure P10.31 about the axle through O if $a = 10.0$ cm and $b = 25.0$ cm.

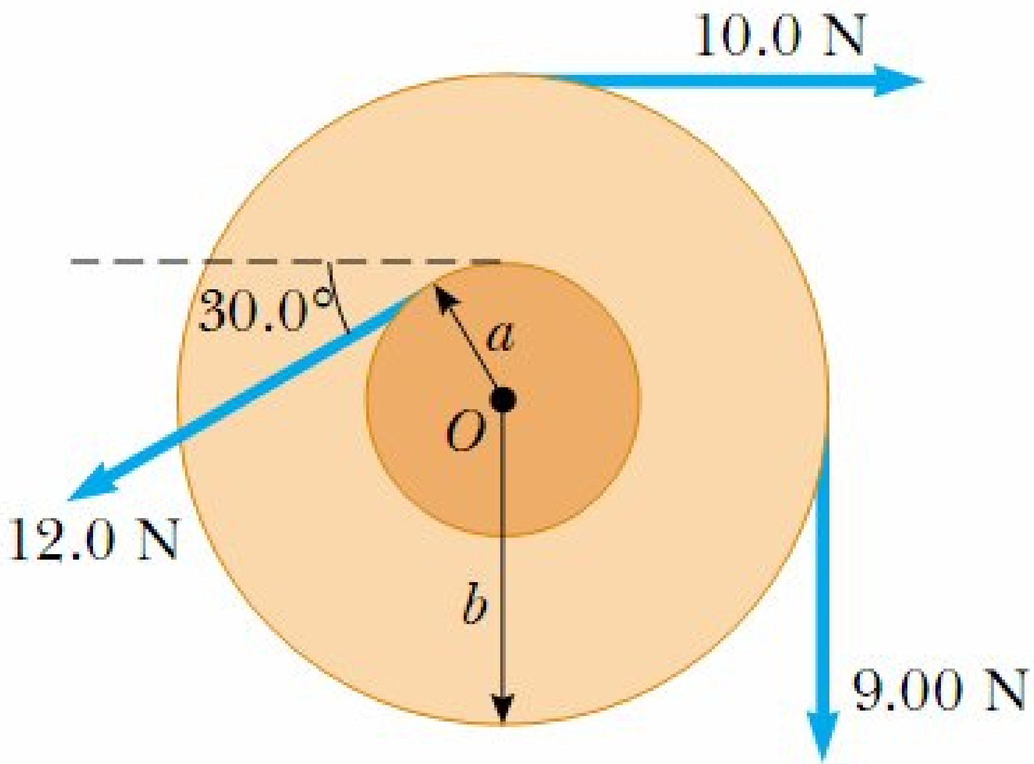
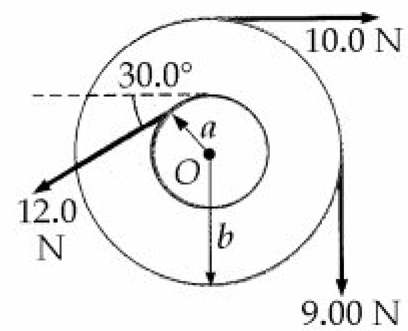


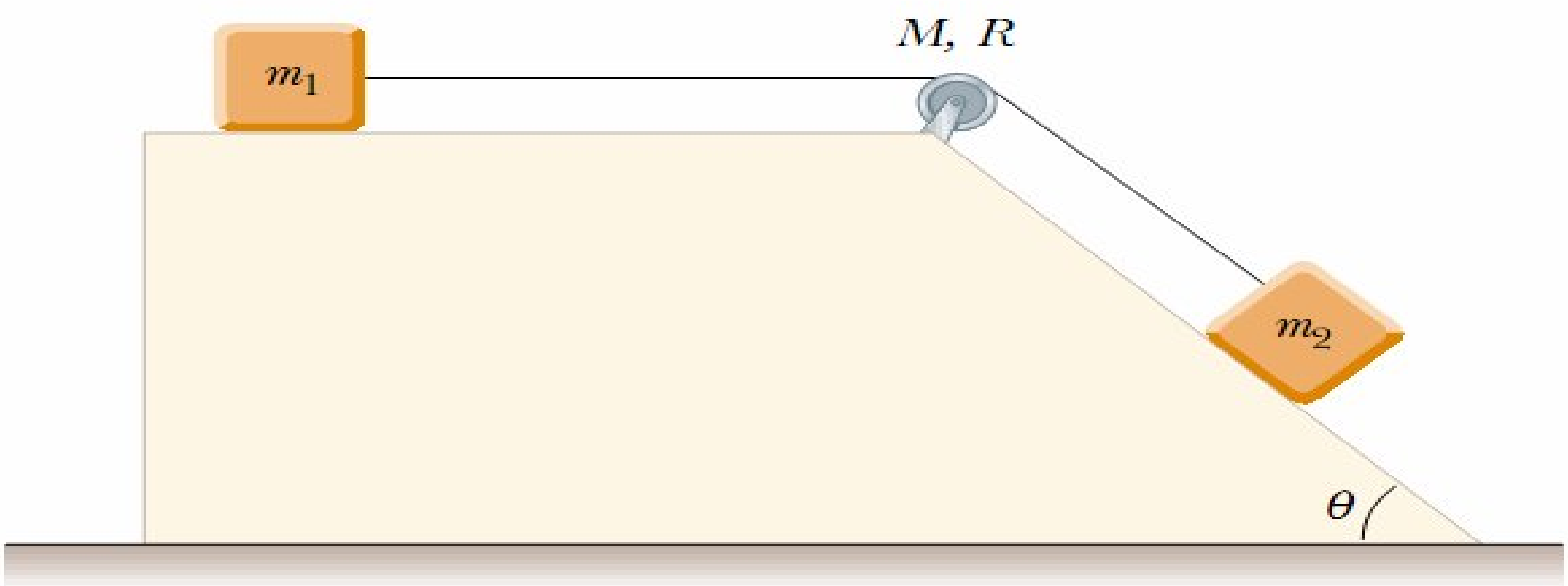
Figure P10.31

$$\sum \tau = 0.100\ \text{m}(12.0\ \text{N}) - 0.250\ \text{m}(9.00\ \text{N}) - 0.250\ \text{m}(10.0\ \text{N}) = \boxed{-3.55\ \text{N}\cdot\text{m}}$$



The thirty-degree angle is unnecessary information.

A block of mass $m_1 = 2.00$ kg and a block of mass $m_2 = 6.00$ kg are connected by a massless string over a pulley in the shape of a solid disk having radius $R = 0.250$ m and mass $M = 10.0$ kg. These blocks are allowed to move on a fixed block-wedge of angle $\theta = 30.0°$ as in Figure P10.37. The coefficient of kinetic friction is 0.360 for both blocks. Draw free-body diagrams of both blocks and of the pulley. Determine (a) the acceleration of the two blocks and (b) the tensions in the string on both sides of the pulley.



For m_1,

$$\sum F_y = ma_y: \quad +n - m_1 g = 0$$

$$n_1 = m_1 g = 19.6\ \text{N}$$

$$f_{k1} = \mu_k n_1 = 7.06\ \text{N}$$

$$\sum F_x = ma_x: \quad -7.06\ \text{N} + T_1 = (2.00\ \text{kg})a \qquad (1)$$

For the pulley,

$$\sum \tau = I\alpha: \quad -T_1 R + T_2 R = \frac{1}{2} MR^2 \left(\frac{a}{R}\right)$$

$$-T_1 + T_2 = \frac{1}{2}(10.0\ \text{kg})a$$

$$-T_1 + T_2 = (5.00\ \text{kg})a \qquad (2)$$

For m_2, $\quad +n_2 - m_2 g \cos\theta = 0$

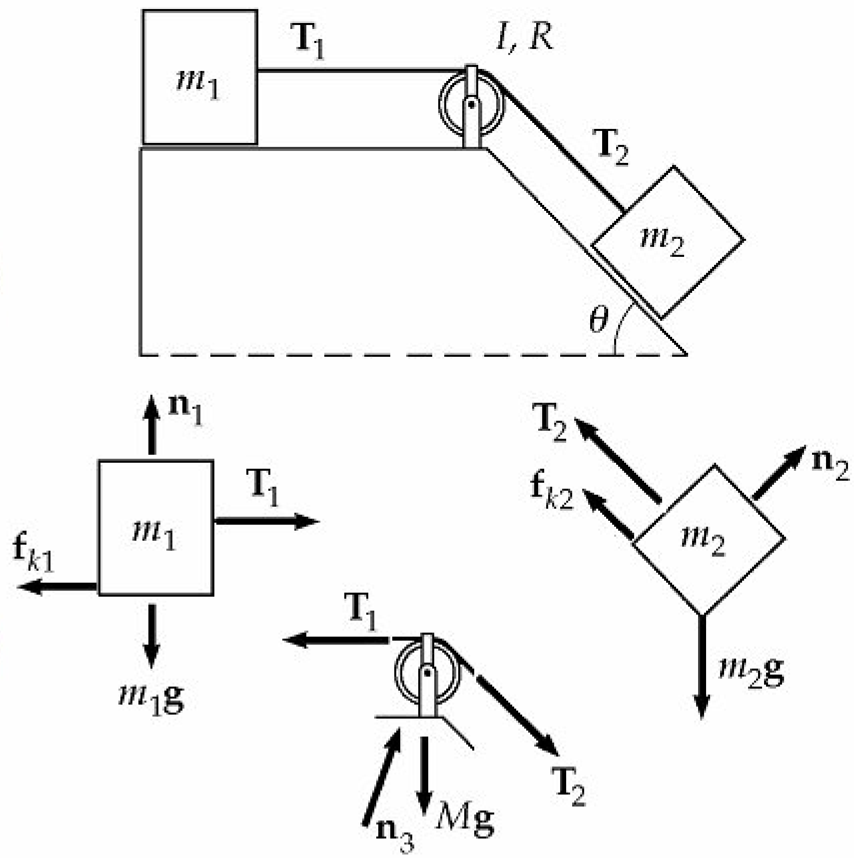
$$n_2 = 6.00\ \text{kg}\left(9.80\ \text{m/s}^2\right)(\cos 30.0°)$$

$$= 50.9\ \text{N}$$

$$f_{k2} = \mu_k n_2$$

$$= 18.3\ \text{N}: \quad -18.3\ \text{N} - T_2 + m_2 \sin\theta = m_2 a$$

$$-18.3\ \text{N} - T_2 + 29.4\ \text{N} = (6.00\ \text{kg})a \qquad (3)$$



(a) Add equations (1), (2), and (3):

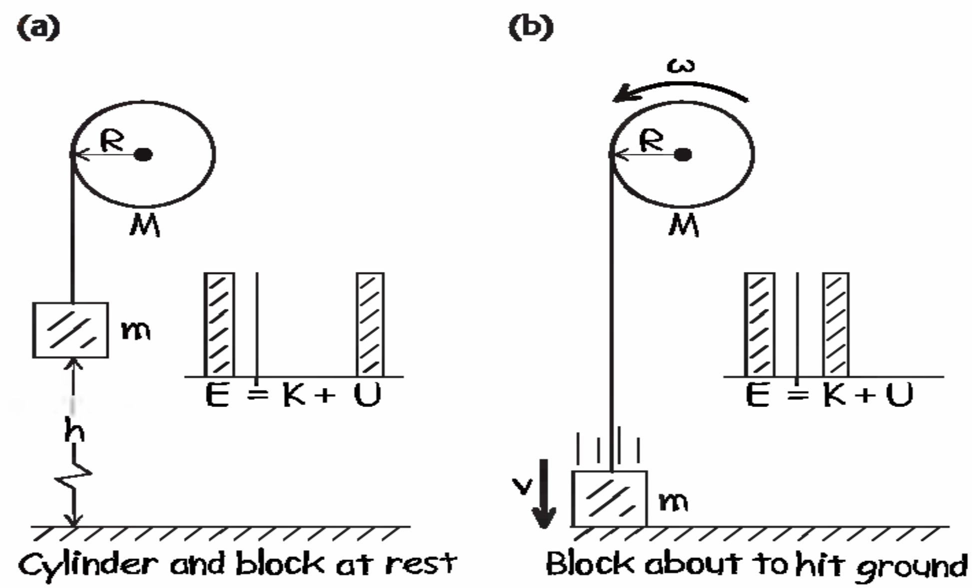
$$-7.06\text{ N} - 18.3\text{ N} + 29.4\text{ N} = (13.0\text{ kg})a$$

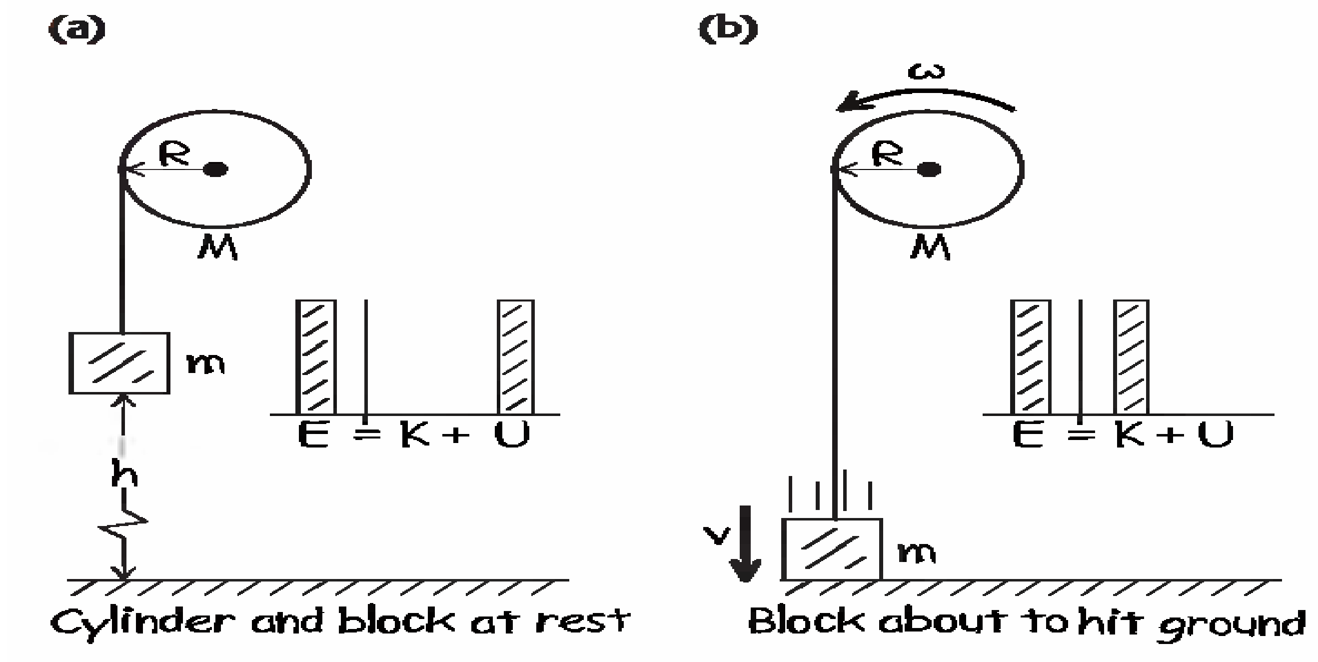
$$a = \frac{4.01\text{ N}}{13.0\text{ kg}} = \boxed{0.309\text{ m/s}^2}$$

(b) $T_1 = 2.00\text{ kg}\left(0.309\text{ m/s}^2\right) + 7.06\text{ N} = \boxed{7.67\text{ N}}$

$T_2 = 7.67\text{ N} + 5.00\text{ kg}\left(0.309\text{ m/s}^2\right) = \boxed{9.22\text{ N}}$

We wrap a light, flexible cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the object with no initial velocity at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping, turning the cylinder. Find the speed of the falling block and the angular speed of the cylinder just as the block strikes the floor.





$$K_1 + U_1 = K_2 + U_2$$

$$0 + mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 + 0 = \frac{1}{2}\left(m + \frac{1}{2}M\right)v^2$$

$$v = \sqrt{\frac{2gh}{1 + M/2m}}$$

The final angular speed of the cylinder is $\omega = v/R$.