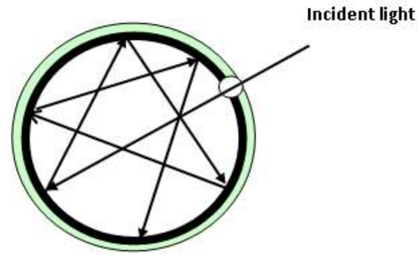
Black Body Radiation

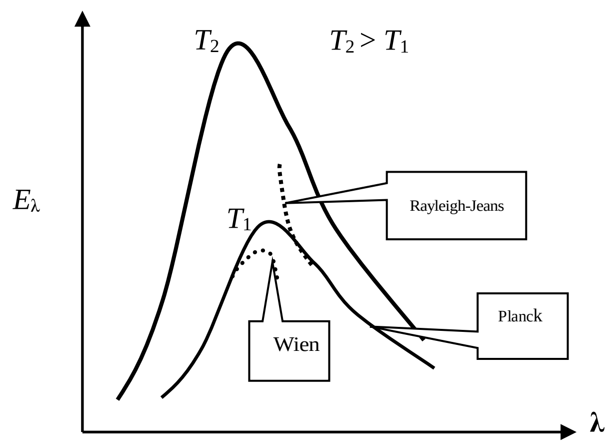
The amount of heat radiation emitted by a body depends on three things:

I- the surface area of the body,
II- the type of surface, and
III- the temperature of the body.



Ideal model for the black body

Comments and laws of black body radiation:



1- Electromagnetic radiation in thermal equilibrium inside an enclosure.
2- Black surfaces are the best emitters and absorbers of radiation at a given temperature.
3- The distribution of the energy flux over the wavelength spectrum does not depend on the nature of the body but does depend on its temperature.
4- The maxima of the curves tend towards short wavelengths at higher temperature.
5- The area between any curve and the wave length axis gives the total energy emitted by the body at that temperature (σT^4) Stefan's law.
6- The curves at lower temperature lie completely inside those of higher temperature.
7- Stefan's law: The total energy flux, φ, (total energy emitted by a black body per unit area of surface per second) is proportional to the fourth power of the body's absolute temperature (T), $\varphi = \sigma T^4$, where $\sigma = 5.67 \times 10^{-8} \ \dfrac{\mathrm{W}}{\mathrm{m^2\ K^4}}$ is the Stefan's constant.
8- Wien's displacement law: $\lambda_{\max} T = 2.9 \times 10^{-3}$ m K. $\lambda_{\max}$ is the wavelength at which most energy is emitted, that is the peak of the curve. Energy emitted at this wavelength is proportional to T^5.

Let us consider the electromagnetic radiation (or in quantum-mechanical language, the assembly of photons) which exists in thermal equilibrium inside an enclosure of volume V whose walls are maintained at absolute temperature T. In this situation photons are continuously absorbed and remitted by the walls; it is, of course, by virtue of these mechanisms that the radiation inside the container depends on the temperature of the walls. The total number of modes (density of quantum states) lying in the momentum range p to $p + dp$ is:

$$g(p)\,\mathrm{d}\,p = \left(\frac{V}{h^3}\right) 4\pi p^2 \,\mathrm{d}\,p \quad \times 2 \text{ polarization states}$$
$$= \left(\frac{V}{h^3}\right) 8\pi p^2 \,\mathrm{d}\,p$$

where the factor of "2" is to take into account the fact that light has two independent directions of polarization.

Using $p = \frac{h\nu}{c}$ and $\mathrm{d}\,p = \frac{h}{c}\mathrm{d}\nu$, then in the frequency range ν to $\nu + d\nu$ we have:

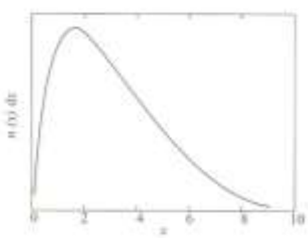
$$g(\nu)d\nu = \left(\frac{V}{c^3}\right) 8\pi\, \nu^2 d\nu \, .$$

where c is the speed of light. Then the Bose-Einstein distribution for photon is:

$$dn(\nu) = n(\nu)d\nu = \frac{g(\nu)d\nu}{e^{\alpha+\beta\varepsilon} - 1} = 8\pi \left(\frac{V}{c^3}\right) \frac{\nu^2 d\nu}{e^{\alpha+\beta\varepsilon} - 1}$$

where $\varepsilon = h\nu$. The requirement that the Lagrangian multiplier $\alpha = 0$ simply means dropping the condition $\delta N = \sum \delta n_i = 0$, for the fixed number of particles. Photons differ from other bosons in that their total number is not conserved.

The number of photons in a given frequency interval is plotted in the figure, where $x = \beta h \nu$. Note that the peak in the number of photons per unit frequency occurs at $x = 1.59$, i.e. at a frequency of $\nu = 1.59/(\beta h)$. This can be easily be deduced by maximizing $n(\nu)$.



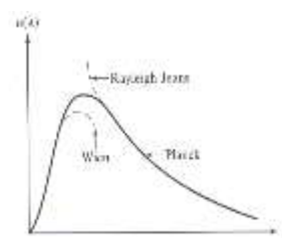
If $dn(\nu)$ multiplied by the energy of photon $\varepsilon = h\nu$, the result is the energy per unit volume, i.e. the energy density in the form:

$$u(\nu)d\nu = h\nu \times \frac{8\pi}{c^3} \frac{\nu^2 d\nu}{e^{\beta h\nu} - 1} = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{\beta h\nu} - 1}, \qquad \text{Planck's law}$$

H.W. Prove that the peak of the above function will be at $\omega = 2.82 \frac{k_B T}{\hbar}$

By using the dimensionless parameter $\eta = \beta h \nu$, i.e. the ratio of photon energy to the thermal energy, one can have:

$$u(\nu)d\nu = h\nu \times 8\pi \frac{\nu^2}{c^3} \frac{d\nu}{e^{\beta h\nu} - 1} = \frac{\hbar}{\pi^2 c^3} \left(\frac{k_B T}{\hbar}\right)^4 \frac{\eta^3 d\eta}{e^{\eta} - 1}$$



Comments:

i- For small frequencies, long wavelength $(\eta << 1, \Rightarrow h\nu << k_B T)$ we have $\left(e^{\beta h\nu} - 1\right) \approx \beta h\nu$ and

$$u(\nu)d\nu = \frac{8\pi k_B T}{c^3} \nu^2 d\nu, \qquad \text{Rayleigh-Jeans law}$$

ii- For high frequencies, short wavelength $(\eta >> 1, \Rightarrow h\nu >> k_B T)$ we have

$$u(\nu)d\nu = \frac{8\pi h}{c^3} \nu^3 e^{-\beta h\nu} d\nu, \qquad \text{Wien's law}$$

iii- maximize using $\frac{du}{d\eta} = \frac{d}{d\eta} \frac{\eta^3 d\eta}{e^{\eta} - 1} = 0$ we have $3e^{\eta} - 3 - \eta e^{\eta} = 0$

By solving the above equation within an approximation $\eta = \beta h \nu_{\max} = \text{constant}$. This implies that $\frac{\nu_{\max}}{T} = \text{constant}$, i.e. $\lambda_{\max} T = \text{constant}$. This is the **Wien displacement law**.

By using the following equation:

$$u(\nu)d\nu = \frac{\hbar}{\pi^2 c^3}\left(\frac{k_B T}{\hbar}\right)^4 \frac{\eta^3 d\eta}{e^\eta - 1}$$

Hence the internal energy per unit volume is:

$$\frac{U}{V} = \int_0^\infty U(\omega,T)d\omega = \frac{\hbar}{\pi^2 c^3}\left(\frac{k_B T}{\hbar}\right)^4 \int_0^\infty \frac{\eta^3}{e^\eta - 1}d\eta$$

$$= \frac{\hbar}{\pi^2 c^3}\left(\frac{k_B T}{\hbar}\right)^4 \left(\frac{\pi^4}{15}\right) = bT^4, \qquad b = \frac{8\pi^5 k_B^4}{15 c^3 h^3} = 7.55\times10^{-16} \text{ J m}^{-3}\text{K}^{-4}$$

This statement is called **Stefan-Boltzmann's law**. Consequently

$$c_v = \left(\frac{\partial u}{\partial T}\right)_V = 4bT^3$$

H.W.: Prove that in case no restrain on n_i, i.e. $\Delta n_i \neq 0$, the partition function is:

$$Z_{\text{photon}} = \prod_{i=1}^{\infty} \frac{1}{1 - e^{\beta\varepsilon_i}}$$

Then,

$$\begin{aligned} F &= -\frac{1}{3}bVT^4, \\ S &= \frac{4}{3}bVT^3, \\ U &= F + TS = bVT^4, \\ P &= \frac{1}{3}bT^4 = \frac{1}{3}\frac{U}{V} \end{aligned} ,$$

Example: The radiation pressure, the mean radiation pressure could be calculated as

$$\bar{P} = \langle P \rangle = \sum_s \bar{n}_s \left(-\frac{\partial \varepsilon_s}{\partial V}\right)$$

using $\epsilon_s = \hbar w = \hbar c\left(\frac{2\pi}{L}\right)\sqrt{(n_x^2 + n_y^2 + n_z^2)} = CV^{-\frac{1}{3}}$, where C is a constant. Then

$$\bar{P} = \sum_s \bar{n}_s\left(-\frac{1}{3}\frac{\epsilon_s}{V}\right) = \frac{1}{3V}\sum_s \bar{n}_s\,\epsilon_s = \frac{\bar{E}}{3V} = \frac{1}{3}\bar{U}$$

The radiation pressure is thus very simple related to the mean energy density of the radiation. It is different than for the Ideal gas where $P = \frac{2}{3}U$.

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$u = aT^4$, $U = bT^4V,$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 4bT^3V,$$

$$S = \int_0^T \frac{đQ}{T} = \int_0^T \frac{C_V dT}{T} = 4bV\int_0^T \frac{T^3 dT}{T} = 4bV\int_0^T T^2 dT = \frac{4}{3}bVT^3$$

$$U = aT^4V \ , \quad S = \frac{4}{3}aVT^3$$

$$F = U - TS = aT^4V - T\,\frac{4}{3}aVT^3 = -\frac{1}{3}aT^4V$$

$$U = aT^4V,\ S = \frac{4}{3}aVT^3,\ F = -\frac{1}{3}aT^4V$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{1}{3}aT^4 = \frac{1}{3}\left(\frac{U}{V}\right) = \frac{1}{3}u$$

Example: calculate the total electromagnetic energy inside an oven of volume 1.0 m^3 heated to a temperature of 400 F

Solution: use the equation $U = bVT^4$, $\quad b = 7.55\times10^{-16}$ J m^{-3}K^{-4}

$$U = bVT^4 = 7.55\times10^{-16}\times(1)\times\left(\frac{400-32}{18}+273\right)^4 = 3.9\times10^{-5}\ \text{J}$$

Example: Show that the thermal energy of the air in the oven is a factor of approximately 10^{10} larger than the electromagnetic energy.

Solution:

$$Thermal\quad energy = \frac{3}{2}nRT = \left(\frac{1000}{22.4}\right)\times 8.314\times10^3\times\left(\frac{400-32}{18}+273\right)$$
$$= 1.77\times10^6\,\text{J}$$

which is 10^{10} times larger than the value of (b)

Example: The radiation pressure at the surface of the sun ($T = 6000\ ^{\circ}\text{C}$) is

$$P = \frac{1}{3}bT^4 = \frac{8\pi^5}{45(ch)^3}(k_BT)^4 = \frac{8\pi^5}{45()}() = ??$$

A standard result in kinetic theory is that the energy flux, φ, through a hole of unit area is:

$$\varphi = \frac{1}{4}\bar{c}\frac{U}{V}$$

where $\bar{c}$ is the mean speed of the particles. And so the total energy flux (i.e. the energy emitted per unit area per unit time by a blackbody) is

$$\varphi = \frac{1}{4}\bar{c}\frac{U}{V} = \frac{1}{4}cbT^4 = \frac{2\pi^5}{15c^2h^3}(k_BT)^4 = \sigma T^4$$

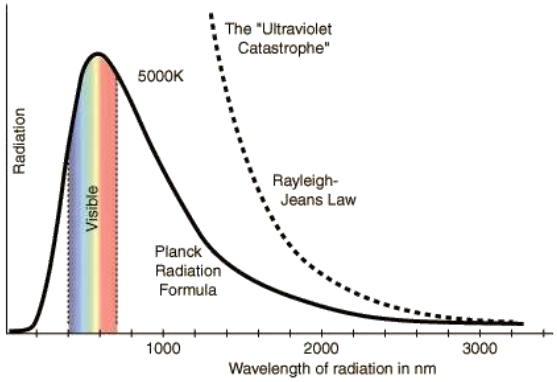
This is known as Stefan's law, and $\sigma = 5.67\times10^{-8}\ \dfrac{\text{W}}{\text{m}^2\ \text{K}^4}$ is Stefan's constant.

Useful Integral and summation:

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}$$

$$\int_0^\infty \frac{x^{1/2}}{e^x-1}dx = 2.612\frac{\sqrt{\pi}}{2}\ , \qquad \int_0^\infty \frac{x^3}{e^x-1}dx = \frac{\pi^4}{15}\ , \qquad \int_0^\infty \frac{x^4 e^x}{\left(e^x-1\right)^2}dx = \frac{4\pi^4}{15}$$

$$\int_0^\infty dx \frac{x^3}{e^x-1} = \sum_{n=1}^{\infty}\int_0^\infty x^3 e^{-nx}dx = \sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^\infty y^3 e^{-y}dy = 6\sum_{n=1}^{\infty}\frac{1}{n^4} = 6\left(\frac{\pi^4}{90}\right) = \frac{\pi^4}{15}.$$



Example: Assume that the radiation from the Sun can be regarded as blackbody radiation. The radiant energy per wavelength interval has a maximum at 480 nm.
(a) Estimate the temperature of the Sun.
(b) Calculate the total radiant power emitted by the Sun. (The radius of the Sun is approximately $7\text{x}10^8$ m.)

a) $\lambda_{\max}T = 2.90\times10^{-3}$ m·K, so $T = \dfrac{2.90\times10^{-3}\ \text{m}\cdot\text{K}}{\lambda_{\max}} = \dfrac{2.90\times10^{-3}\ \text{m}\cdot\text{K}}{4.80\times10^{-7}\ \text{m}}$

$\underline{T = 6.04\times10^3\ \text{K} = 6040\ \text{K}}$

b) $P = \sigma T^4 A = \sigma T^4 4\pi R^2 = (5.67\times10^{-8}\ \text{W}\cdot\text{m}^{-2}\text{K}^{-4})(6040\ \text{K})^4 4\pi(7.0\times10^8\ \text{m})^2$

$\underline{P = 4.65\times10^{26}\ \text{W}}$

Helmholtz Free Energy

As long as we don't want any detailed information about the microstates of a system, we can determine its properties from the thermodynamic potentials instead of from the partition function. In particular we can use the Helmholtz free energy.

Using: $Z_{\text{photons}} = \prod_{i=1}^{\infty} \frac{1}{1-\mathrm{e}^{-\beta\varepsilon_i}}$, then

$$F = -k_B T \ln Z_{\text{photons}}$$

$$= k_B T \sum_{\text{states, } r=1}^{\infty} \ln(1-\mathrm{e}^{-\beta\varepsilon_r}) \equiv k_B T \int_{\text{levels, } i=1}^{\infty} \rho(\varepsilon_i) \ln(1-\mathrm{e}^{-\beta\varepsilon_i})\, \mathrm{d}\varepsilon_i$$

Using $\rho(\omega)\,\mathrm{d}\omega = V\frac{\omega^2}{\pi^2 \mathrm{c}^3}\mathrm{d}\omega$, we have

$$F = k_B T \int_0^{\infty} \frac{V\omega^2}{\pi^2 \mathrm{c}^3} \ln(1-\mathrm{e}^{-\beta\hbar\omega})\,\mathrm{d}\omega$$

Letting $x = \beta\hbar\omega, \quad \mathrm{d}x = \beta\hbar\,\mathrm{d}\omega$, we get

$$F = k_B T \frac{V}{\pi^2 \mathrm{c}^3}\left(\frac{k_B T}{\hbar}\right)^3 \underbrace{\int_0^{\infty} x^2 \ln(1-\mathrm{e}^{-x})\,\mathrm{d}x}_{-\frac{\pi^2}{45}} = k_B T \frac{V}{\pi^2 \mathrm{c}^3}\left(\frac{k_B T}{\hbar}\right)^3 \times -\frac{\pi^2}{45}$$

$$= -\frac{1}{3} bVT^4$$

Entropy

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = -\frac{\partial}{\partial T}\left(-\frac{1}{3}bVT^4\right) = \frac{4}{3}bVT^3$$

Pressure

$$P = -\left(\frac{\partial F}{\partial V}\right)_T = -\frac{\partial}{\partial V}\left(-\frac{1}{3}bVT^4\right) = \frac{1}{3}bT^4$$

Internal energy

$$U = F - TS = bVT^4$$

Pressure: the pressure can then be rewritten

$$P = \frac{1}{3}\frac{U}{V}$$

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Show that for an ideal boson gas $P = \alpha\, u$, where P is the pressure, u the energy density and α a constant. What is α?

Answer

Using results in section 7.H.1d., we have

$$\langle E\rangle = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} \langle n_{\mathbf{k}}\rangle = \sum_{\mathbf{k}} \frac{\varepsilon_{\mathbf{k}}\, z}{e^{\beta\varepsilon_{\mathbf{k}}} - z} = \frac{4\pi V}{(2\pi\hbar)^3}\int_0^\infty dp\ p^2\, \frac{1}{2m}\, p^2\, \frac{z}{e^{\left(\beta\frac{1}{2m}p^2\right)} - z}$$

Using $p = \sqrt{2mk_BT}\ x\ = \dfrac{2\sqrt{\pi}\ \hbar}{\lambda_T} x$, we have,

$$\begin{aligned}
U = \langle E\rangle &= \frac{4\pi V}{(2\pi\hbar)^3}\cdot\frac{1}{2m}\left(\frac{2\sqrt{\pi}\ \hbar}{\lambda_T}\right)^5 \int_0^\infty dx\ x^4\, \frac{z}{\exp\left(x^2\right) - z} = \frac{8V\hbar^2\sqrt{\pi}}{m\lambda_T^5}\int_0^\infty dx\ x^4\, \frac{z}{\exp\left(x^2\right) - z} \\
&= \frac{8V\hbar^2\sqrt{\pi}}{m\lambda_T^5} K_4^{(-)}(z) = \frac{8V\hbar^2\sqrt{\pi}}{m\lambda_T^5}\cdot\frac{1}{2}\Gamma\left(\frac{5}{2}\right) g_{5/2}(z) \\
&= \frac{3V\hbar^2\pi}{m\lambda_T^5}\, g_{5/2}(z)
\end{aligned}$$

$$\Rightarrow \qquad u = \frac{U}{V} = \frac{3\hbar^2\pi}{m\lambda_T^5}\, g_{5/2}$$

Comparing with

$$P = k_B T\, \frac{1}{\lambda_T^3}\, g_{5/2}(z) = \frac{2\hbar^2\pi}{m\lambda_T^5}\, g_{5/2}(z)$$

we have

$$P = \frac{2}{3} u$$

Consider the photons in equilibrium inside a cubic box with volume $V = L^3$ and temperature T at the walls. The photon energies are $\hbar\omega_i = \hbar c k_i$, where k_i is the wavevector of the ith standing wave. Compute pressure P of this photon gas.

Answer

The allowed photon states are standing waves that vanish at the walls. Thus,

$$\mathbf{k} = \frac{\pi}{L}\left(n_x, n_y, n_z\right) \qquad \text{with} \qquad n_j = 1, 2, \cdots \qquad \text{for} \qquad j = x, y, z$$

so that

$$\omega = ck = \frac{c\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} \qquad (3)$$

and

$$\sum_{\mathbf{k}} \rightarrow \left(\frac{L}{\pi}\right)^3 \int_{k_x, k_y, k_z > 0} d^3k\ = \left(\frac{L}{2\pi}\right)^3 \int d^3k$$

Since there are 2 transverse modes for each **k**, the sum over modes is

$$\sum_{\alpha=1,2}\sum_{\mathbf{k}} f(\mathbf{k},\alpha) \to \left(\frac{L}{2\pi}\right)^3 \sum_{\alpha=1,2}\int d^3k\, f(\mathbf{k},\alpha)$$

When *f* is independent of polarization, we have

$$\sum_{\alpha=1,2}\sum_{\mathbf{k}} f(\mathbf{k}) = 2\sum_{\mathbf{k}} f(\mathbf{k}) \to 2\left(\frac{L}{2\pi}\right)^3 \int d^3k\, f(\mathbf{k})$$

Since $\mu' = 0$, the grand partition function is

$$Z_\mu(T) = Tr\left[\exp\left(-\beta H\right)\right] = \left(\prod_{\mathbf{k},\alpha}\sum_{n_\mathbf{k}=0}^{\infty}\right)\exp\left(-\beta\sum_{\mathbf{k},\alpha} n_\mathbf{k}\hbar\omega_\mathbf{k}\right)$$

$$= \prod_{\mathbf{k},\alpha}\left[\sum_{n_\mathbf{k}=0}^{\infty}\exp\left(-\beta n_\mathbf{k}\hbar\omega_\mathbf{k}\right)\right] = \prod_{\mathbf{k},\alpha}\left[\frac{1}{1-\exp\left(-\beta\hbar\omega_\mathbf{k}\right)}\right]$$

$$= \prod_{\mathbf{k}}\left[\frac{1}{1-\exp\left(-\beta\hbar\omega_\mathbf{k}\right)}\right]^2 \tag{1}$$

which gives a grand potential

$$\Omega(T,V,\mu) = -k_B T \ln Z_\mu(T) = k_B T\sum_{\mathbf{k},\alpha}\ln\left[1-\exp\left(-\beta\hbar\omega_\mathbf{k}\right)\right] \tag{2}$$

$$= 2k_B T\left(\frac{L}{2\pi}\right)^3\int d^3k \ln\left[1-\exp\left(-\beta\hbar\omega_\mathbf{k}\right)\right]$$

Using

$$\int d^3k\, f(k) = 4\pi\int dk\, k^2 f(k) = \frac{4\pi}{c^3}\int d\omega\,\omega^2 f\left(\frac{\omega}{c}\right)$$

we have

$$\Omega = \pi k_B T\left(\frac{L}{\pi c}\right)^3\int_0^\infty d\omega\,\omega^2\ln\left[1-\exp\left(-\beta\hbar\omega\right)\right] \tag{2a}$$

Now,

$$I = \int_0^\infty d\omega\,\omega^2\ln\left[1-\exp\left(-\beta\hbar\omega\right)\right] = \frac{1}{3}\int_0^\infty d\omega^3\ \ln\left[1-\exp\left(-\beta\hbar\omega\right)\right]$$

$$= \frac{1}{3}\omega^3\ln\left[1-\exp\left(-\beta\hbar\omega\right)\right]\Big|_0^\infty - \frac{1}{3}\int_0^\infty d\omega\,\omega^3\frac{\beta\hbar\exp\left(-\beta\hbar\omega\right)}{1-\exp\left(-\beta\hbar\omega\right)} \tag{2b}$$

The 1st term involves the function

$$f(x) = x\ln\left(1-e^{-x}\right)$$

In particular

$$f(\infty) = \lim_{x\to\infty}\left[x\left(-e^{-x}-\frac{1}{2}e^{-2x}-\cdots\right)\right] = 0$$

where we've used

$$\ln(1+x) = x - \frac{1}{2}x^2 + \cdots$$

and, by repeated application of the L'Hospital rule,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'}{g'}$$

to get

$$\lim_{x\to\infty}x^n e^{-x}=\lim_{x\to\infty}\frac{x^n}{e^x}=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}=\lim_{x\to\infty}\frac{n!}{e^x}=0$$

Also

$$\begin{aligned}
f(0)&=\lim_{x\to0}\left[x\ln\left(-\sum_{n=1}^{\infty}(-)^n\frac{x^n}{n!}\right)\right]=\lim_{x\to0}\left\{x\ln\left[x\left(1-\sum_{n=2}^{\infty}(-)^n\frac{x^{n-1}}{n!}\right)\right]\right\}\\
&=\lim_{x\to0}\left\{x\ln x+x\ln\left(1-\sum_{n=2}^{\infty}(-)^n\frac{x^{n-1}}{n!}\right)\right\}\\
&=\lim_{x\to0}\left\{x\ln x+x\ln\left(1-\frac{1}{2}x+\frac{1}{3!}x^2-\cdots\right)\right\}\\
&=\lim_{x\to0}(x\ln x)=\lim_{x\to0}\frac{\ln x}{x^{-1}}=-\lim_{x\to0}\frac{x^{-1}}{x^{-2}}=-\lim_{x\to0}x=0
\end{aligned}$$

Hence, (2b) becomes

$$I=-\frac{1}{3}\int_0^\infty d\omega\,\omega^3\frac{\beta\hbar\exp(-\beta\hbar\omega)}{1-\exp(-\beta\hbar\omega)}=-\frac{1}{3(\beta\hbar)^3}\int_0^\infty dx\,x^3\frac{e^{-x}}{1-e^{-x}}\qquad(2c)$$

Now,

$$\begin{aligned}
J_n&=\int_0^\infty dx\,x^n\frac{1}{e^x-1}=\int_0^\infty dx\,x^n\frac{e^{-x}}{1-e^{-x}}=\int_0^\infty dx\,x^n e^{-x}\sum_{m=0}^{\infty}e^{-m\,x}\\
&=\sum_{m=0}^{\infty}\int_0^\infty dx\,x^n e^{-(m+1)x}=\sum_{m=0}^{\infty}\frac{1}{(m+1)^{n+1}}\int_0^\infty dy\,y^n e^{-y}\\
&=n!\sum_{m=0}^{\infty}\frac{1}{(m+1)^{n+1}}=n!\sum_{m=1}^{\infty}\frac{1}{m^{n+1}}=n!\,\zeta(n+1)
\end{aligned}$$

where ζ is the Reimann zeta function. Thus, (2c) becomes

$$\begin{aligned}
I&=-\frac{1}{3(\beta\hbar)^3}J_3=-\frac{1}{3(\beta\hbar)^3}3!\,\zeta(4)=-\frac{1}{3(\beta\hbar)^3}3!\,\zeta(4)\\
&=-\frac{1}{3(\beta\hbar)^3}3!\left(\frac{\pi^4}{90}\right)=-\frac{\pi^4}{45(\beta\hbar)^3}
\end{aligned}$$

Hence, eq(2a) is

$$\Omega=\pi k_BT\left(\frac{L}{\pi c}\right)^3 I=-\pi k_BT\left(\frac{L}{\pi c}\right)^3\frac{\pi^4}{45(\beta\hbar)^3}=-(k_BT)^4\left(\frac{L}{\hbar c}\right)^3\frac{\pi^2}{45}$$

The pressure is

$$P=-\frac{\Omega}{V}=(k_BT)^4\left(\frac{1}{\hbar c}\right)^3\frac{\pi^2}{45}=\frac{1}{3}\sigma T^4\qquad(6)$$

where $\sigma = k_B \left(\frac{1}{\hbar c}\right)^3 \frac{\pi^2}{15}$ is the **Stefan- Boltzmann constant**.

Photons Sec 7.4

a. Photon gas

Consider electromagnetic radiation inside a box. We may regard the electromagnetic field as a superposition of standing waves that fit between the walls of the box. The system, then, consists of these standing waves, rather than literal atoms oscillating back and forth. The energy of a standing wave of a particular frequency, *f*, is quantized $E_i = ihf$. Therefore, the sum over states for a single frequency is $Z = \sum_{i=0}^{\infty} e^{-\frac{ihf}{kT}} = \frac{1}{1-e^{-\frac{hf}{kT}}}$. The average energy associated with that frequency is

$$< E(f) > = -\frac{1}{Z}\frac{\partial Z}{\partial (kT)} = \frac{hf}{e^{\frac{hf}{kT}}-1}.$$

This expression is known as the *Planck Distribution*. Since each quantum of EM energy is *hf*, the average number of photons of frequency *f* is

$$n_{Pl} = \frac{1}{e^{\frac{hf}{kT}}-1}.$$

In effect, we are treating the system as a gas composed of photons.

b. Total energy

We have the average energy of photons of frequency *f* in the box. The total energy contained in the box is obtained by summing over the allowed frequencies. The frequencies are restricted by the finite volume of the box. For instance, along the *x*-axis, the frequencies of standing waves that will fit in the length L_x are $f_{n_x} = \frac{n_x c}{2L_x}$. The corresponding energies are $E_{n_x} = hf_{n_x} = \frac{n_x hc}{2L_x}$. Of course, the same applies to the *y*- and *z*-axes. In three dimensions, $E_n = \frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2}$. (Let's say the box is a cube of side *L*.) The total energy of the photons in the box is

$$U = 2\sum_{n_x,n_y,n_z} \frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2}\,\frac{1}{e^{\frac{E_n}{kT}}-1}.$$

We are adding up the points in a spherical volume of radius $n = \sqrt{n_x^2+n_y^2+n_z^2}$. Since the number of photons in the box is very, very large at any but very low temperatures, the sum can go over to an integral.

$$U = \int_0^{\infty} n^2 \frac{hcn}{L}\frac{1}{e^{\frac{hcn}{2LkT}}-1}dn \int_0^{\pi/2}\sin\theta d\theta \int_0^{\pi/2} d\phi = \frac{\pi}{2}\frac{hc}{L}\int_0^{\infty}\frac{n^3}{e^{\frac{hcn}{2LkT}}-1}dn$$

Now let us change variables from *n* to E_n.

$$E_n = \frac{hcn}{2L}, \text{ whence } dn = \frac{2L}{hc}dE \text{ and } n^3 = \left(\frac{2L}{hc}E_n\right)^3.$$

$$U = \frac{\pi}{2}\frac{hc}{L}\int_0^\infty \frac{n^3}{e^{\frac{hcn}{2LkT}}-1}dn = \frac{8\pi L^3}{(hc)^3}\int_0^\infty \frac{E^3}{e^{\frac{E}{kT}}-1}dE$$

$$\frac{U}{L^3} = \frac{U}{V} = \frac{8\pi}{(hc)^3}\int_0^\infty \frac{E^3}{e^{\frac{E}{kT}}-1}dE$$

$$\frac{U}{V} = \frac{8\pi^5 (kT)^4}{15(hc)^3}$$

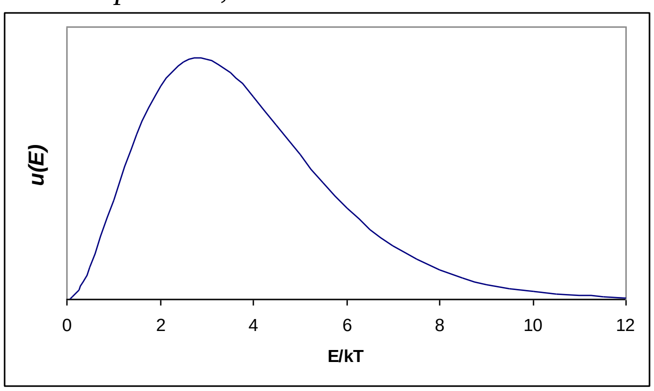
We can now compute also the heat capacity and entropy of the photon gas.

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 4\frac{8\pi^5 k^4 L^3}{15(hc)^3}T^3$$

$$S(T) = \int_0^T \frac{C_V(T')}{T'}dT' = \frac{32\pi^5}{45}V\left(\frac{kT}{hc}\right)^3 k$$

c. Black body spectrum

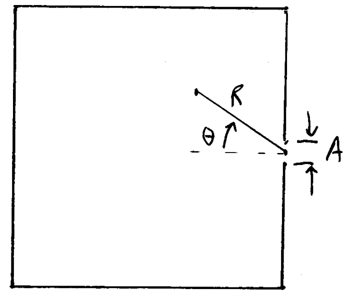
What we have here is the *energy per unit volume per unit energy*, $u(E) = \frac{8\pi}{(hc)^3}\frac{E^3}{e^{\frac{E}{kT}}-1}$, also called the spectrum of the photons. It's named the *Planck spectrum*, after the fellow who first worked it out, Max Planck.



Notice that $\frac{U}{V} \propto T^4$, and that the spectrum peaks at $E = 2.82kT$. These "Laws" had been obtained empirically, and called Stefan-Boltzmann's "Law" and Wein's Displacement "Law."

d. Black body radiation

Of course, the experimentalists were measuring the spectra of radiation from various material bodies at various temperatures. Perhaps we should verify that the radiation emitted by a material object is the same as the spectrum of photon energies in the oven. So, consider an oven at temperature *T*, and imagine a small hole in one side. What is the spectrum of photons that escape through that hole? Well, the spectrum of the escaping photons must be the same as the photon gas in the oven, since all photons travel at the same speed, *c*. By a similar token, the energy emitted through the hole is proportional to T^4.



Finally, we might consider a perfectly absorbing material object exchanging energy by radiation with the hole in the oven. In equilibrium (at the same *T* as the oven), the material object (the black body) must radiate the same power and spectrum as the hole, else they would be violating the Second "Law" of thermodynamics.