

KING FAHD UNIVERSITY OF PETROLUIM AND MINERALS
DEPARTMENT OF PHYSICS
PHYS 530 STATISTICAL PHYSICS
FALL 2005
FIRST MAJOR(18/9/2005)
TIME (9:00 P.M. ----- 11:00 A.M.)

Answer the following problems. (SHOW YOUR WORK)

1- Consider a hypothetical system of 3 states.

First state is non-degenerate with an energy $E_o = 0$ and occupation number $n_o = 0$.

Second state has an energy $E_1 = \Delta > 0$ and occupation number $n_1 = 1$. Finally, the third two-fold degenerate state has an energy $E_2 = 2\Delta$ and occupation number $n_2 = 2$.

$$n_2 = 2 \text{=====} E_2 = 2\Delta$$

$$n_1 = 1 \text{-----} E_1 = \Delta$$

$$n_o = 0 \text{-----} E_o = 0$$

i- Calculate the partition function.

ii- If the average occupation number $\langle n \rangle = 1$, compute the temperature of the system (in terms of Δ)

2- Consider a system of N free and distinguish particles in which the energy of each particle can assume two and only two distinct values, 0 and E ($E > 0$). Denote by n_o and n_1 the occupation numbers of the energy level 0 and E , respectively. The total energy of the system is U .

i- Find the total number of microstates $w(N, n_o, n_1)$ and obtain the entropy of such a system as a function of n_o and n_1 .

ii- Express n_o , n_1 and the entropy as a function of the energy U .

iii- Find the temperature as a function of the energy U , and show that the condition of

having $T < 0$ is $\frac{U}{NE} < \frac{1}{2}$.

- 3- Consider a system of N classical anharmonic oscillators with a restoring force proportional to the cube of the displacement ($F = -Kx^3$, instead of the usual linear restoring force) enclosed in a large volume.
- Do you have to consider the Gibb's correction factor? Are the oscillators distinguishable or indistinguishable?
 - If the single particle partition function of the system has the form $Z_{sp} = A\beta^m$, calculate the value of the constants A and m .
 - Calculate the pressure, the internal energy, the entropy and the specific heat.
- 4- An ideal gas, which consists of N indistinguishable atoms of mass m is located inside an infinitely high cylinder of radius R . In the presence of uniform downward gravitational field, find the
- partition function,
 - free energy and
 - the specific heat of this system.

Solve only one of the following two problems

- 5- The molecules of an imaginary ideal gas have internal energy levels that are equally spaced so that the n th energy eigen value is:

$$E_n = n\varepsilon, \quad n = 0, 1, 2, \dots$$

The degeneracy of the n^{th} energy level is $n + 1$.

- Calculate the total partition function of the system.
 - Calculate the average energy.
- 6- A classical N indistinguishable particles obey the relation:

$$\varepsilon = c |p|,$$

Where c is a constant and positive, p is their momentum and ε is their energy.

- Calculate the total partition function of the system.
- Calculate the free energy, and
- Calculate the specific heat

Physics 530 Final exam
Formula sheet
Fall Semester 2005-2006 (Term 051)

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}, \quad T = \left(\frac{\partial F}{\partial S}\right)_{V,N}, \quad P = -\left(\frac{\partial F}{\partial V}\right)_{N,T}, \quad C_v = \left(\frac{\partial U}{\partial T}\right)_{N,V}$$

$$S = Nk \left[\ln\left(\frac{V}{N}\right) + \frac{3}{2} \ln\left(\frac{2\pi mkT}{h^2}\right) + \frac{5}{2} \right] \quad g(\varepsilon)d\varepsilon = g_s \frac{2\pi V}{h^3} (2m)^{3/2} \varepsilon^{1/2} d\varepsilon$$

$$S = k \ln w, \quad P_r = \frac{1}{Z} e^{-\beta E_r}, \quad \langle n_i \rangle = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial \varepsilon_i}, \quad \langle \varepsilon \rangle = -\frac{\partial \ln Z}{\partial \beta}, \quad Z_{sp} = \sum_{n=0}^{\infty} g_n e^{-\beta \varepsilon_n}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad -1 < x < 1$$

Stirling's formula: $N! \approx N \ln N - N$

Integrals

$$\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}}, \quad n > 1, 2, 3, \dots, \quad a > 0$$

$$\int_0^{\infty} e^{-ax} dx = \frac{1}{a}, \quad \int_0^{\infty} x e^{-ax} dx = \frac{1}{a^2}, \quad \int_0^{\infty} x^2 e^{-ax} dx = \frac{2}{a^3}, \quad \int_{-\infty}^{\infty} e^{-ax^4} dx = \frac{2\Gamma(5/2)}{a^{1/4}}$$

$$\int_0^{\infty} x^{2n+1} e^{-ax^2} dx = \frac{n!}{2a^{n+1}}, \quad n > 1, 2, 3, \dots, \quad a > 0$$

$$\int_0^{\infty} x^{2n} e^{-ax^2} dx = \frac{1.3.5 \dots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}}, \quad n > 1, 2, 3, \dots, \quad a > 0$$

$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$	$\int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a}$
$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a} \sqrt{\frac{\pi}{a}}$	$\int_0^{\infty} x^3 e^{-ax^2} dx = \frac{1}{2a^2}$
$\int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{3}{4a^2} \sqrt{\frac{\pi}{a}}$	$\int_0^{\infty} x^5 e^{-ax^2} dx = \frac{1}{a^3}$

1-i-
$$Z_{sp} = \sum_{n=0}^{\infty} g_i e^{-\beta \varepsilon_i} = 1 + e^{-\beta \Delta} + 2e^{-2\beta \Delta}$$

ii-

$$\begin{aligned} \langle n \rangle &= \sum_i n_i P_i = \frac{0 + 1 \times e^{-\beta \Delta} + 2 \times 2e^{-2\beta \Delta}}{1 + e^{-\beta \Delta} + 2e^{-2\beta \Delta}} \\ &= \frac{x + 4x^2}{1 + x + 2x^2}, \quad x = e^{-\beta \Delta} \end{aligned}$$

With the condition $\langle n \rangle = 1$

$$\begin{aligned} \Rightarrow \frac{x + 4x^2}{1 + x + 2x^2} &= 1 \\ \Rightarrow x = e^{-\beta \Delta} &= \frac{1}{\sqrt{2}} \\ \Rightarrow k_B T &= \frac{2}{\ln 2} \Delta \approx 2.885 \Delta \end{aligned}$$

2- a- Use the definition:

$$\left. \begin{array}{l} n_o N \equiv \text{partilcles with energy } 0 \\ n_1 N \equiv \text{partilcles with energy } E \end{array} \right\} n_o + n_1 = N$$

and

$$w(N, n_o, n_1) = \frac{N!}{(n_o N)!(n_1 N)!}, \quad S = k_B \ln w$$

With Stirling's formula:

$$S = -Nk_B \{n_o \ln n_o + n_1 \ln n_1\}$$

b- Average energy per particle:

$$\begin{aligned} \frac{U}{N} = En_1 \quad \text{or} \quad n_1 &= \frac{U}{NE} \\ S &= -Nk_B \left\{ \frac{U}{NE} \ln \left(\frac{U}{NE} \right) + \left(1 - \frac{U}{NE} \right) \ln \left(1 - \frac{U}{NE} \right) \right\} \end{aligned}$$

c-

$$\begin{aligned}\frac{1}{T} &= \frac{\partial S}{\partial U} = \frac{1}{NE} \frac{\partial S}{\partial \left(\frac{U}{NE}\right)} = -\frac{k_B}{E} \left\{ \ln\left(\frac{U}{NE}\right) - \ln\left(1 - \frac{U}{NE}\right) \right\} \\ &= \frac{k_B}{E} \ln\left(\frac{1 - \frac{U}{NE}}{\frac{U}{NE}}\right)\end{aligned}$$

or

$$U = \frac{NE}{e^{\beta E} + 1}$$

For

$$T < 0, \quad \Rightarrow \quad \frac{U}{NE} < \frac{1}{2}$$

3- Localized particles are distinguishable, which implies no Gibbs' correction factor.

$$\begin{aligned}Z_{sp} &= \frac{1}{h} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq e^{-\beta \frac{p^2}{2m} - \beta \frac{Kq^4}{4}} \\ &= \frac{1}{h} \underbrace{\int_{-\infty}^{\infty} dp e^{-\beta \frac{p^2}{2m}}}_{\sqrt{\frac{2\pi}{\beta h^2}}} \underbrace{\int_{-\infty}^{\infty} dq e^{-\beta \frac{Kq^4}{4}}}_{\left(\frac{4}{\beta K}\right)^{1/4} \int_{-\infty}^{\infty} dx e^{-x^4}} \\ &= A \beta^{-3/4}, \quad A = \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \left(\frac{4}{K}\right)^{1/4} \sqrt{\frac{2m\pi}{h^2}}\end{aligned}$$

The free energy is

$$F = -k_B T \ln Z_N = N k_B T \left[\frac{3}{4} \ln \beta - \ln A \right]$$

$$P = -\left(\frac{\partial A}{\partial V}\right)_T = 0$$

$$S = -\left(\frac{\partial A}{\partial T}\right)_N = N k_B \left[\frac{3}{4} \ln \beta + \ln A + \frac{3}{4} \right]$$

$$U = F + TS = \frac{3}{4} N k_B T$$

$$C_V = -T \left(\frac{\partial^2 F}{\partial T^2}\right)_V = \frac{3}{4} N k_B$$

4- An ideal gas, which consists of N indistinguishable atoms of mass m is located inside an infinitely high cylinder of radius R . In the presence of uniform downward gravitational field, find the

i- partition function,

ii- free energy and

iii-specific heat of this system.

$$\begin{aligned} Z_{sp} &= \frac{1}{h^3} \int d^3r d^3p e^{-\beta \frac{p^2}{2m} - \beta mg} \\ &= \frac{\pi R^2}{h^3} \underbrace{\left(\int_0^\infty dz e^{-\beta mg} \right)}_{\frac{1}{\beta mg}} \underbrace{\left(\int_{-\infty}^\infty dp_x e^{-\beta \frac{p_x^2}{2m}} \right)^3}_{\left(\frac{2\pi m}{\beta} \right)^{3/2}} \\ &= \frac{R^2}{h^3} \frac{\sqrt{m}}{2} \left(\frac{2\pi}{\beta} \right)^{5/2} \end{aligned}$$

Where R is the radius of the cylinder. The full partition function of the N indistinguishable atoms is $Z_N = \frac{1}{N!} (Z_{sp})^N$. The free energy is

$$F = -k_B T \ln Z_N \approx Nk_B T [\ln N - 1 - \ln Z_{sp}]$$

And the specific heat is:

$$C_V = -T \left(\frac{\partial^2 F}{\partial T^2} \right)_V = \frac{5}{2} N$$

5- The molecules of an imaginary ideal gas have internal energy levels that are equally spaced so that the n th energy eigen value is:

$$E_n = n\varepsilon, \quad n = 0, 1, 2, \dots$$

i- Calculate the total partition function of the system.

$$\begin{aligned} Z_{sp} &= \sum_{n=0}^{\infty} g_n e^{-\beta \varepsilon_n} = \sum_{n=0}^{\infty} (n+1) e^{-\beta \varepsilon n} = \sum_{n=0}^{\infty} (n+1) x^n, \quad x = e^{-\beta \varepsilon} \\ &= \frac{d}{dx} \sum_{m=0}^{\infty} x^m, \quad m = n+1, \\ &= \frac{d}{dx} (1-x)^{-1} = (1-x)^{-2} = (1-e^{-\beta \varepsilon})^{-2} \end{aligned}$$

ii- Calculate the average energy.

$$\langle E_i \rangle = -\frac{\partial \ln Z}{\partial \beta} = \frac{2\varepsilon}{e^{-\beta\varepsilon} - 1}$$

6- A classical N indistinguishable particles obey the relation:

$$\varepsilon = c |p|,$$

where $c > 1$ is a constant, p is their momentum and ε is their energy. For the proposed system calculate

- i- the total partition function,
- ii- the free energy, and
- iii- the specific heat.

i-

$$\begin{aligned} Z_{sp} &= \frac{1}{h^3} \underbrace{\int d^3 r}_V \int d^3 p e^{-\beta cp} = \frac{V}{h^3} 4\pi \left(\int_0^\infty dp p^2 e^{-\beta cp} \right) \\ &= \frac{4\pi V}{h^3} \frac{\partial^2}{\partial(\beta c)^2} \underbrace{\left(\int_0^\infty dp p^2 e^{-\beta cp} \right)}_{\left(\frac{1}{\beta c}\right)} = \frac{8\pi V}{h^3} \left(\frac{1}{\beta c}\right)^3 \end{aligned}$$

Then the total partition function is:

$$Z_N = \frac{1}{N!} Z_{sp}^N$$

ii- Free energy

$$F = -k_B T \ln Z_N = k_B T (N \ln N - N) - N k_B T \ln Z_{sp}$$

iv- The specific heat is:

$$\begin{aligned} C_v &= \left(\frac{\partial^2 F}{\partial T^2} \right)_{N,V} = -N k_B T \frac{\partial^2}{\partial T^2} (-3T \ln T) \\ &= 3N k_B \end{aligned}$$