KING FAHD UNIVERSITY OF PETROLUIM AND MINERALS
DEPARTMENT OF PHYSICS
<u>PHYS 530</u> <u>STATISTICAL PHYSICS</u>
FALL 2005
FIRST MAJOR(18/9/2005)
TIME (9:00 P.M. ----- 11:00 A.M.)

<u>Answer the following problems. (SHOW YOUR WORK)</u>

1- Consider a hypothetical system of 3 states. First state is non-degenerate with an energy $E_o = 0$ and occupation number $n_o = 0$. Second state has an energy $E_1 = \Delta > 0$ and occupation number $n_1 = 1$. Finaly, the third two-fold degenerate state has an energy $E_2 = 2\Delta$ and occupation number $n_2 = 2$.

$n_2 = 2$ $\qquad$ $E_2 = 2\Delta$

$n_1 = 1$ $\qquad$ $E_1 = \Delta$

$n_o = 0$ $\qquad$ $E_o = 0$

i- Calculate the partition function.

ii- If the average occupation number $\langle n \rangle = 1$, compute the temperature of the system (in terms of Δ)

2- Consider a system of N free and distinguish particles in which the energy of each particle can assume two and only two distinct values, 0 and $E\,(E > 0)$. Denote by n_o and n_1 the occupation numbers of the energy level 0 and E, respectively. The total energy of the system is U.

i- Find the total number of microstates $w\,(N, n_o, n_1)$ and obtain the entropy of such a system as a function of n_o and n_1.

ii- Express n_o, n_1 and the entropy as a function of the energy U.

iii-Find the temperature as a function of the energy U, and show that the condition of having $T < 0$ is $\dfrac{U}{NE} < \dfrac{1}{2}$.

3- Consider a system of N classical anharmonic oscillators with a restoring force proportional to the cube of the displacement ($F = -Kx^3$, instead of the usual linear restoring force) enclosed in a large volume.

i- Do you have to consider the Gibb's correction factor? Are the oscillators distinguishable or indistinguishable?

ii- If the single particle partition function of the system has the form $Z_{sp} = A\beta^m$, calculate the value of the constants A and m.

iii- Calculate the pressure, the internal energy, the entropy and the specific heat.

4- An ideal gas, which consists of N indistinguishable atoms of mass m is located inside an infinitely high cylinder of radius R. In the presence of uniform downward gravitational field, find the

i- partition function,

ii- free energy and

iii-the specific heat of this system.

<u>Solve only one of the following two problems</u>

5- The molecules of an imaginary ideal gas have internal energy levels that are equally spaced so that the nth energy eigen value is:

$$E_n = n\varepsilon, \qquad n = 0,1,2,\dots$$

The degeneracy of the n^{th} energy level is $n+1$.

i- Calculate the total partition function of the system.

ii- Calculate the average energy.

6- A classical N indistinguishable particles obey the relation:

$$\varepsilon = c\,|p|,$$

Where c is a constant and positive, p is their momentum and ε is their energy.

i- Calculate the total partition function of the system.

ii- Calculate the free energy, and

iii- Calculate the specific heat

Physics 530 Final exam

Formula sheet

Fall Semester 2005-2006 (Term 051)

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}, \quad T = \left(\frac{\partial F}{\partial S}\right)_{V,N}, \quad P = -\left(\frac{\partial F}{\partial V}\right)_{N,T}, \quad C_v = \left(\frac{\partial U}{\partial T}\right)_{N,V}$$

$$S = Nk\left[\ln\left(\frac{V}{N}\right) + \frac{3}{2}\ln\left(\frac{2\pi mkT}{h^2}\right) + \frac{5}{2}\right] \qquad g(\varepsilon)d\varepsilon = g_s \frac{2\pi V}{h^3}(2m)^{3/2}\varepsilon^{1/2}d\varepsilon$$

$$S = k\ln w, \quad P_r = \frac{1}{Z}e^{-\beta E_{er}}, \quad < n_i > = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_i} \quad < \varepsilon > = -\frac{\partial \ln Z}{\partial \beta} \quad Z_{sp} = \sum_{n=0}^{\infty} g_i e^{-\beta\varepsilon_i}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad -1 < x < 1$$

Stirling's formula: $N! = N\ln N - N$

Integrals

$$\int_0^\infty x^n e^{-ax}dx = \frac{n!}{a^{n+1}}, \qquad n > 1, 2, 3, \cdots, \quad a > 0$$

$$\int_0^\infty e^{-ax}dx = \frac{1}{a}, \quad \int_0^\infty x e^{-ax}dx = \frac{1}{a^2}, \quad \int_0^\infty x^2 e^{-ax}dx = \frac{2}{a^3}, \quad \int_{-\infty}^\infty e^{-ax^4}dx = \frac{2\Gamma(5/2)}{a^{1/4}}$$

$$\int_0^\infty x^{2n+1} e^{-ax^2}dx = \frac{n!}{2a^{n+1}}, \qquad n > 1, 2, 3, \cdots, \quad a > 0$$

$$\int_0^\infty x^{2n} e^{-ax^2}dx = \frac{1.3.5....(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}. \qquad n > 1, 2, 3, \cdots, \quad a > 0$$

$\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$	$\int_0^\infty x e^{-ax^2}dx = \frac{1}{2a}$
$\int_{-\infty}^\infty x^2 e^{-ax^2}dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$	$\int_0^\infty x^3 e^{-ax^2}dx = \frac{1}{2a^2}$
$\int_{-\infty}^\infty x^4 e^{-ax^2}dx = \frac{3}{4a^2}\sqrt{\frac{\pi}{a}}$	$\int_0^\infty x^5 e^{-ax^2}dx = \frac{1}{a^3}$

1-i-

$$Z_{sp} = \sum_{n=0}^{\infty} g_i e^{-\beta\varepsilon_i} = 1 + e^{-\beta\Delta} + 2e^{-2\beta\Delta}$$

ii-

$$\langle n \rangle = \sum_i n_i P_i = \frac{0 + 1\times e^{-\beta\Delta} + 2\times 2e^{-2\beta\Delta}}{1 + e^{-\beta\Delta} + 2e^{-2\beta\Delta}}$$

$$= \frac{x + 4x^2}{1 + x + 2x^2}, \qquad x = e^{-\beta\Delta}$$

With the condition $\langle n \rangle = 1$

$$\Rightarrow \quad \frac{x + 4x^2}{1 + x + 2x^2} = 1$$

$$\Rightarrow x = e^{-\beta\Delta} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad k_B T = \frac{2}{\ln 2}\Delta \approx 2.885\Delta$$

1- a- Use the definition:

$$\left.\begin{array}{l} n_o N \equiv \text{partilcles with energy } 0 \\ n_1 N \equiv \text{partilcles with energy } E \end{array}\right\} n_o + n_1 = N$$

and

$$w(N, n_o, n_1) = \frac{N!}{(n_o N)!(n_1 N)!}, \qquad S = k_B \ln w$$

With Stirling’s formula:

$$S = -Nk_B \{n_o \ln n_o + n_1 \ln n_1\}$$

b- Average energy per particle:

$$\frac{U}{N} = En_1 \quad \text{or} \quad n_1 = \frac{U}{NE}$$

$$S = -Nk_B \left\{ \frac{U}{NE} \ln\left(\frac{U}{NE}\right) + \left(1 - \frac{U}{NE}\right) \ln\left(1 - \frac{U}{NE}\right) \right\}$$

c-

$$\frac{1}{T} = \frac{\partial S}{\partial U} = \frac{1}{NE}\frac{\partial S}{\partial\left(\frac{U}{NE}\right)} = -\frac{k_B}{E}\left\{\ln\left(\frac{U}{NE}\right) - \ln\left(1 - \frac{U}{NE}\right)\right\}$$
$$= \frac{k_B}{E}\ln\left(\frac{1-\frac{U}{NE}}{\frac{U}{NE}}\right)$$

or

$$U = \frac{NE}{e^{\beta E} + 1}$$

For

$$T < 0, \quad \Rightarrow \quad \frac{U}{NE} < \frac{1}{2}$$

2- Localized particles are distinguishable, which implies no Gibbs' correction factor.

$$Z_{sp} = \frac{1}{h}\int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dq e^{-\beta\frac{p^2}{2m}-\beta\frac{Kq^4}{4}}$$

$$= \underbrace{\frac{1}{h}\int_{-\infty}^{\infty} dp e^{-\beta\frac{p^2}{2m}}}_{\sqrt{\frac{2\pi}{\beta h^2}}} \underbrace{\int_{-\infty}^{\infty} dq e^{-\beta\frac{Kq^4}{4}}}_{\left(\frac{4}{\beta K}\right)^{1/4}\int_{-\infty}^{\infty} dx\, e^{-x^4}}$$

$$= A\beta^{-3/4}, \qquad A = \tfrac{1}{2}\Gamma\left(\tfrac{1}{4}\right)\left(\frac{4}{K}\right)^{1/4}\sqrt{\frac{2m\pi}{h^2}}$$

The free energy is

$$F = -k_B T \ln Z_N = Nk_B T\left[\tfrac{3}{4}\ln\beta - \ln A\right]$$

$$P = -\left(\frac{\partial A}{\partial V}\right)_T = 0$$

$$S = -\left(\frac{\partial A}{\partial T}\right)_N = Nk_B\left[\tfrac{3}{4}\ln\beta + \ln A + \tfrac{3}{4}\right]$$

$$U = F + TS = \frac{3}{4}Nk_B T$$

$$C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V = \frac{3}{4}Nk_B$$

And the specific heat is:

$$C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V = \frac{5}{2}N$$

3- An ideal gas, which consists of N indistinguishable atoms of mass m is located inside an infinitely high cylinder of radius R. In the presence of uniform downward gravitational field, find the

i- partition function,

ii- free energy and

iii-specific heat of this system.

$$Z_{sp} = \frac{1}{h^3}\int d^3r d^3p e^{-\beta\frac{p^2}{2m}-\beta mg}$$

$$= \frac{\pi R^2}{h^3}\underbrace{\left(\int_0^\infty dz\, e^{-\beta mg}\right)}_{\frac{1}{\beta mg}}\underbrace{\left(\int_{-\infty}^{\infty} dp_x\, e^{-\beta\frac{p_x^2}{2m}}\right)^3}_{\left(\frac{2\pi m}{\beta}\right)^{3/2}}$$

$$= \frac{R^2}{h^3}\frac{\sqrt{m}}{2}\left(\frac{2\pi}{\beta}\right)^{5/2}$$

Where R is the radius of the cylinder. The full partition function of the N indistinguishable atoms is $Z_N = \frac{1}{N!}\left(Z_{sp}\right)^N$. The free energy is

$$F = -k_BT\,\ln Z_N \approx Nk_BT\left[\ln N - 1 - \ln Z_{sp}\right]$$

And the specific heat is:

$$C_V = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_V = \frac{5}{2}N$$

4- The molecules of an imaginary ideal gas have internal energy levels that are equally spaced so that the nth energy eigen value is:

$$E_n = n\varepsilon, \qquad n = 0,1,2,\ldots$$

i- Calculate the total partition function of the system.

$$Z_{sp} = \sum_{n=0}^{\infty} g_i e^{-\beta\varepsilon_i} = \sum_{n=0}^{\infty}(n+1)e^{-\beta\varepsilon n} = \sum_{n=0}^{\infty}(n+1)x^n, \quad x = e^{-\beta\varepsilon}$$

$$= \frac{d}{dx}\sum_{m=0}^{\infty} x^m, \qquad m = n+1,$$

$$= \frac{d}{dx}(1-x)^{-1} = (1-x)^{-2} = (1-e^{-\beta\varepsilon})^{-2}$$

ii- Calculate the average energy.

$$< E_i > = -\frac{\partial \ln Z}{\partial \beta} = \frac{2\varepsilon}{e^{-\beta\varepsilon}-1}$$

5- A classical indistinguishable particles obey the relation:

$$\varepsilon = c\left|p\right|,$$

where $c > 1$ is a constant, p is their momentum and ε is their energy. For the proposed system calculate

i- the total partition function,

ii- the free energy, and

iii- the specific heat.

i-

$$Z_{sp} = \frac{1}{h^3}\underbrace{\int d^3r}_{V}\int d^3p\, e^{-\beta cp} = \frac{V}{h^3}4\pi\left(\int_0^\infty dpp^2 e^{-\beta cp}\right)$$

$$= \frac{4\pi V}{h^3}\frac{\partial^2}{\partial(\beta c)^2}\underbrace{\left(\int_0^\infty dpp^2 e^{-\beta cp}\right)}_{\left(\frac{1}{\beta c}\right)} = \frac{8\pi V}{h^3}\left(\frac{1}{\beta c}\right)^3$$

Then the total partition function is:

$$Z_N = \frac{1}{N!}Z_{sp}^N$$

ii- Free energy

$$F = -k_BT\,\ln Z_N = k_BT\,(N\,\ln N - N) - Nk_BT\,\ln Z_{sp}$$

iv-The specific heat is:

$$C_v = \left(\frac{\partial^2 F}{\partial T^2}\right)_{N,V} = -Nk_BT\,\frac{\partial^2}{\partial T^2}(-3T\,\ln T)$$

$$= 3Nk_B$$