Instructor Prof. Dr. I. Nasser

KING FAHD UNIVERSITY of PETROLIUM and MINERALS
Physics Department
Quantum Mechanics I (Phys-401)
T-082

Issued: 23-5-2009	Second Major	Time: 3- hours

A- Give a brief but reasoned answer to each of the following:

i- Prove that: $\left\langle \hat{L}_x^2 \right\rangle = \frac{1}{2}\left[\ell(\ell+1)\hbar^2 - m^2\hbar^2 \right]$

ii- Express the function $|\psi> = -3\sin^2\theta + 2.$ in Cartesian coordinates.

iii- An electron in a hydrogen atom is in the (normalized) state:

$$\psi(\boldsymbol{r}) = \frac{1}{6}\left[4|1,0,0\rangle + 3|2,1,1\rangle - i\,|2,1,0\rangle + \sqrt{10}\,|3,0,0\rangle \right]$$

What is the probability that a measurement of the electron's energy will yield the values of -3.4 eV ?

iv- If the electron in a hydrogen atom is in the state: $\psi(\boldsymbol{r}) = A\left(\frac{1}{\pi^3}\right)^{1/4} e^{-r^2/2}\cos\theta$. Can a measurement of the electron's energy yield the value -13.6 eV ? Why or Why not?

v- Is the function $\sin\theta(1-\cos\theta)(\cos\varphi + i\,\sin\varphi)$ eigenfunction of L^2 ? or L_z ? or both?

vi- Calculate the value of the matrix $<Y_{3,2}|\hat{L}_x|Y_{3,1}>$

vii- Calculate the matrix $<Y_{3,-2}|\hat{L}_-\hat{L}_+ - \hat{L}_+\hat{L}_-|Y_{3,-2}>$ **.**

viii- Use the state function $|1,1> = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$, to calculate the state function $|1,0>$.

ix- Suppose that the wave function for the 2s state of Hydrogen atom is given by

$$\psi_{2s}(r) = N(1+br)e^{-r/2}$$

Find the constant *b*. [Note that: $\psi_{1s}(r) = \frac{1}{\pi}e^{-r}$]

x- Consider the normalized function: $\psi = -\frac{1}{8\sqrt{\pi}}(x+iy)e^{-r/2}$; find $|n,\ell,m>$ and the most probable value of r .

xi- What is the probability that an electron in the 1s orbital will be within a radius of 1.5 Bohr radius ?

xii- Find the expression of $\hat{L}_y\left|Y_{1,0}\right\rangle$.

(5-points each, total of 60 points)

Instructor Prof. Dr. I. Nasser

Solve the following problem and show your work

4- Consider a system which is described by the state:

$$|\psi\rangle = \sqrt{\frac{3}{8}}Y_{1,1} + \sqrt{\frac{1}{8}}Y_{1,0} + AY_{1,-1}$$

where A is a real constant.

a- Calculate A so that $|\psi\rangle$ is normalized. (5-points)

b- Find $\hat{L}_+|\psi\rangle$. (5-points)

c- Calculate the expectation values of $\hat{L}_x$ and $\hat{L}^2$ in the state $|\psi\rangle$. (10-points)

d- Find the probability associated with a measurement that gives zero for z- component of the angular momentum. (5-points)

e- Calculate $\langle\varphi|\hat{L}_z|\psi\rangle$ and $\langle\varphi|\hat{L}_-|\psi\rangle$ where (10-points)

$$|\varphi\rangle = \sqrt{\frac{8}{15}}Y_{2,1} + \sqrt{\frac{4}{15}}Y_{1,0} + \sqrt{\frac{3}{15}}Y_{2,-1}$$

(Total of 45 points)

B- Solve only two problems and show your work

1- An electron in a hydrogen atom is in a state given by the wave function:

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{3\pi}} re^{-r} \quad .$$

a- Find the value of ℓ for this state. Justify your answer. (5-points)

b- If $\psi(\boldsymbol{r})$ is expanded in hydrogen eigenstates as:

$$\psi(\boldsymbol{r}) = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} C_{n\ell m}\psi_{n\ell m}(\boldsymbol{r})$$

for which values of n, ℓ and m will be $C_{n\ell m}$ non-zero. . (5-points)

c- What is the probability that a measurement of the energy of this electron will give the value -13.6 eV ? . (15-points)

(Total of 25 points)

2- An electron in the Coulomb field of a proton is in a state described by the following spatial wave function

$$\psi(r) = A\left[3|1,0,0\rangle + 2|2,1,1\rangle - |2,1,0\rangle + \sqrt{10}|3,1,-1\rangle\right]$$

a- Find the value of A.

b- What is the expectation value of the energy?

c- What is the expectation value of the L^2?

d- What is the expectation value of the L_z?

e- What is the expectation value of the L_+?

Instructor Prof. Dr. I. Nasser

(5-points each, total of 25 points)

3- A particle in the state:

$$\psi(x, y, z) = \sqrt{\frac{3}{8\pi}}\left(\frac{z - iy}{r}\right)$$

a- Write the above expression in terms of spherical harmonics. . (15-points)
b- What is the expectation value of the L^2? (2-points)
c- Suppose L_z is measured. What are the possible results? (1-point)
d- What is the probability of obtaining each result? (5-points)
e- What is the expectation value of the L_z? (2-points)
f- Find $\langle \hat{L}_- \rangle$. (5-points)

(Total of 30 points)

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C- Give a brief but reasoned answer to each of the following:

xiii- Prove that: $\left\langle \hat{L}_x^2 \right\rangle = \frac{1}{2}\left[\ell(\ell+1)\hbar^2 - m^2\hbar^2\right]$

Answer:

$$\hat{L}_x = \frac{1}{2}\left(\hat{L}_+ + \hat{L}_-\right) \Rightarrow \hat{L}_x^2 = \frac{1}{4}\left(\hat{L}_+ + \hat{L}_-\right)^2 = \frac{1}{4}\left(\hat{L}_+^2 + \hat{L}_-^2 + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right)$$

$$\left\langle \hat{L}_x^2 \right\rangle = \left\langle \ell m \left| \hat{L}_x^2 \right| \ell m \right\rangle = \frac{1}{4}\left\langle \ell m \right|\left(\cancel{\hat{L}_+^2} + \cancel{\hat{L}_-^2} + \hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right)\left| \ell m \right\rangle$$

$$= \frac{1}{4}\left\langle \ell m \right|\left(\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+\right)\left| \ell m \right\rangle, \qquad \hat{L}_+^\dagger = \hat{L}_-$$

$$= \frac{1}{4}\left(\left|C_-\right|^2 + \left|C_+\right|^2\right) = \frac{1}{2}\left[\ell(\ell+1)\hbar^2 - m^2\hbar^2\right]$$

xiv- Express the function $|\psi> = -3\sin^2\theta + 2.$ in Cartesian coordinates.

Answer: $|\psi> = (3z^2 - r^2)/r^2$

xv- An electron in a hydrogen atom is in the (normalized) state:

$$\psi(\boldsymbol{r}) = \frac{1}{6}\left[4\left|1,0,0\right\rangle + 3\left|2,1,1\right\rangle - i\left|2,1,0\right\rangle + \sqrt{10}\left|3,0,0\right\rangle\right]$$

What is the probability that a measurement of the electron's energy will yield the values of -3.4 eV ?

Answer: This energy corresponds to n = 2:, so, $P(-3.4) = \left|\frac{3}{6}\right|^2 + \left|\frac{-i}{6}\right|^2 = \frac{5}{18}$.

xvi- If the electron in a hydrogen atom is in the state: $\psi(\boldsymbol{r}) = A\left(\frac{1}{\pi^3}\right)^{1/4} e^{-r^2/2}\cos\theta$. Can a measurement of the electron's energy yield the value -13.6 eV ? Why or Why not?

Answer: This state corresponds to l = 1, m=0, so a measurement of the energy cannot give -13.6 eV .

xvii- Is the function $\sin\theta(1-\cos\theta)(\cos\varphi + i\sin\varphi)$ eigenfunction of L^2 ? or L_z ? or both?

Answer: $\sin\theta(1-\cos\theta)(\cos\varphi + i\sin\varphi) = -\sqrt{\frac{8\pi}{3}}Y_{1,1} - \sqrt{\frac{8\pi}{15}}Y_{2,1}$, so it is eigenfunction of L_z .

xviii-Calculate the value of the matrix $<Y_{3,2}|\hat{L}_x|Y_{3,1}>$.

Answer: Use $L_x = (L_+ + L_-)/2$, then

$$<Y_{3,2}|\hat{L}_x|Y_{3,1}>=\frac{1}{2}\left(<Y_{3,2}|\hat{L}_+|Y_{3,1}>+<Y_{3,2}|\hat{L}_-|Y_{3,1}>\right)$$

$$=\frac{1}{2}\left(<Y_{3,2}|C_+|Y_{3,2}>+\underbrace{<Y_{3,2}|C_-|Y_{3,0}>}_{=\delta_{3,3}\delta_{2,-1}=0}\right)=\frac{C_+}{2}$$

with l = 3 and m=1, we can have

$$C_+=\hbar\sqrt{l(l+1)-m(m+1)}=\hbar\sqrt{3(4)-1(2)}=\hbar\sqrt{10} \quad \Rightarrow \quad <Y_{3,2}|\hat{L}_x|Y_{3,1}>=\hbar\frac{\sqrt{10}}{2}$$

xix- Calculate the matrix $<Y_{3,-2}|\hat{L}_-\hat{L}_+ - \hat{L}_+\hat{L}_-|Y_{3,-2}>$.

Answer:

Using
$$\left.\begin{array}{l}\hat{L}_-\hat{L}_+=\hat{L}^2-\hat{L}_z^2-\hbar\hat{L}_z\\ \hat{L}_+\hat{L}_-=\hat{L}^2-\hat{L}_z^2+\hbar\hat{L}_z\end{array}\right\}\hat{L}_-\hat{L}_+-\hat{L}_+\hat{L}_-=-2\hbar\hat{L}_z$$

Then

$$\left\langle Y_{3,-2}\middle|\hat{L}_-\hat{L}_+-\hat{L}_+\hat{L}_-\middle|Y_{3,-2}\right\rangle=\left\langle Y_{3,-2}\middle|-2\hbar\hat{L}_z\middle|Y_{3,-2}\right\rangle=-2\hbar(-2\hbar)=4\hbar^2$$

xx- Use the state function $|1,1>=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}$, to calculate the state function $|1,0>$.

Answer:

first, use $\hat{L}_-|1,1>=C_-|1,0>=\sqrt{2}\hbar|1,0>$, (1)

second use,

$$\hat{L}_-|1,1>=\hbar e^{-i\varphi}\left[\frac{\partial}{\partial\theta}-i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right]\left(-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\varphi}\right)$$

$$=\hbar e^{-i\varphi}\left[\sqrt{\frac{3}{8\pi}}\cos\theta e^{i\varphi}+\sqrt{\frac{3}{8\pi}}\cos\theta e^{i\varphi}\right] \quad (2)$$

$$=2\hbar\sqrt{\frac{3}{8\pi}}\cos\theta$$

(1)=(2) $\Rightarrow$ $|1,0>=-\hbar\sqrt{\frac{3}{4\pi}}\cos\theta$

xxi- Suppose that the wave function for the 2s state of Hydrogen atom is given by

$$\psi_{2s}(r) = N(1+br)e^{-r/2}$$

Find the constant *b*. [Note that: $\psi_{1s}(r) = \frac{1}{\pi}e^{-r}$]

Answer: From the orthogonality condition between the two states, one finds b = -1/2.

xxii- Consider the normalized function: $\psi = -\frac{1}{8\sqrt{\pi}}(x+iy)e^{-r/2}$; find $|n,\ell,m>$ and the most probable value of r .

Answer: $\psi = -A(x+iy)e^{-r/2} = A\sqrt{\frac{8\pi}{3}}\, rY_{1,1}e^{-r/2}$, $\Rightarrow$ $|n,l,m> \equiv |2,1,1>$

$$\frac{\partial}{\partial r}\left(r\ re^{-r/2}\right) = 0 \Rightarrow r = 4.$$

xxiii-What is the probability that an electron in the 1s orbital will be within a radius of 1.5 Bohr radius ?

Answer: $P = \int |\psi|^2 r^2 dr = 4\pi\int_0^{1.5}\left(\frac{e^{-r}}{\sqrt{\pi}}\right)^2 r^2 dr = 0.58 \Rightarrow 58\%$

xxiv- Find the expression of $\hat{L}_y\left|Y_{1,0}\right\rangle$.

Answer: $\hat{L}_y\left|Y_{1,0}\right\rangle = -\frac{i}{\sqrt{2}}\hbar\left(Y_{1,1} - Y_{1,-1}\right)$

D- Solve the following problem and show your work

5- Consider a system which is described by the state:

$$|\psi\rangle = \sqrt{\frac{3}{8}}Y_{1,1} + \sqrt{\frac{1}{8}}Y_{1,0} + AY_{1,-1}$$

where *A* is a real constant.

f- Calculate A so that $|\psi\rangle$ is normalized.

g- Find $\hat{L}_+|\psi\rangle$.

h- Calculate the expectation values of $\hat{L}_x$ and $\hat{L}^2$ in the state $|\psi\rangle$.

i- Find the probability associated with a measurement that gives zero for z- component of the angular momentum.

j- Calculate $\langle\varphi|\hat{L}_z|\psi\rangle$ and $\langle\varphi|\hat{L}_-|\psi\rangle$ where

$$|\varphi\rangle = \sqrt{\frac{8}{15}}Y_{2,1} + \sqrt{\frac{4}{15}}Y_{1,0} + \sqrt{\frac{3}{15}}Y_{2,-1}$$

a) Calculate A so that $|\psi\rangle$ is normalized. **Ans** $A = 1/\sqrt{2}$

b) Find $\hat{L}_+|\psi\rangle$. **Ans** $\hat{L}_+|\psi\rangle = \hbar\left[\frac{1}{2}|1,1\rangle + |1,0\rangle\right]$

c) Calculate the expectation values of $\hat{L}_x$ and $\hat{L}^2$ in the state $|\psi\rangle$.

Ans:

$$\langle\psi|\hat{L}^2|\psi\rangle = \left\langle\left(\sqrt{\frac{3}{8}}Y_{1,1} + \sqrt{\frac{1}{8}}Y_{1,0} + \sqrt{\frac{1}{2}}Y_{1,-1}\right)\middle|\hat{L}^2\middle|\left(\sqrt{\frac{3}{8}}Y_{1,1} + \sqrt{\frac{1}{8}}Y_{1,0} + \sqrt{\frac{1}{2}}Y_{1,-1}\right)\right\rangle$$

$$= \hbar^2\left\langle\left(\sqrt{\frac{3}{8}}Y_{1,1} + \sqrt{\frac{1}{8}}Y_{1,0} + \sqrt{\frac{1}{2}}Y_{1,-1}\right)\middle\|\left(2\sqrt{\frac{3}{8}}Y_{1,1} + 2\sqrt{\frac{1}{8}}Y_{1,0} + 2\sqrt{\frac{1}{2}}Y_{1,-1}\right)\right\rangle$$

$$= \left(2\frac{3}{8} + 2\frac{1}{8} + 2\frac{1}{2}\right)\hbar^2 = 2\hbar^2$$

$$\langle\psi|\hat{L}_x|\psi\rangle = \frac{\sqrt{2}}{8}\hbar\left[\sqrt{3}+2\right]$$

Ans: $\hat{L}_x|\psi\rangle = \left(\frac{\hat{L}_+ + \hat{L}_-}{2}\right)\left(\sqrt{\frac{3}{8}}Y_{1,1} + \sqrt{\frac{1}{8}}Y_{1,0} + \sqrt{\frac{1}{2}}Y_{1,-1}\right)$

$$\hat{L}_+\left(\sqrt{\frac{3}{8}}Y_{1,1} + \sqrt{\frac{1}{8}}Y_{1,0} + \sqrt{\frac{1}{2}}Y_{1,-1}\right) = \hbar\left(0\times\sqrt{\frac{3}{8}}Y_{1,2} + \sqrt{2}\times\sqrt{\frac{1}{8}}Y_{1,1} + \sqrt{2}\times\sqrt{\frac{1}{2}}Y_{1,0}\right)$$

$$\langle\psi|\hat{L}_+|\psi\rangle = \hbar\left\langle\left(\sqrt{\frac{3}{8}}Y_{1,1}+\sqrt{\frac{1}{8}}Y_{1,0}+\sqrt{\frac{1}{2}}Y_{1,-1}\right)\middle|\sqrt{2}\times\sqrt{\frac{1}{8}}Y_{1,1}+\sqrt{2}\times\sqrt{\frac{1}{2}}Y_{1,0}\right\rangle$$

$$=\hbar\sqrt{2}\left(\frac{\sqrt{3}}{8}+\frac{1}{4}\right)=\hbar\frac{\sqrt{2}}{8}\left(\sqrt{3}+2\right)$$

Also, $\langle\psi|\hat{L}_-|\psi\rangle = \hbar\frac{\sqrt{2}}{8}\left(\sqrt{3}+2\right)$

$$\left\langle\hat{L}_x\right\rangle = \left\langle\left(\frac{\hat{L}_+ + \hat{L}_-}{2}\right)\right\rangle = \hbar\frac{\sqrt{2}}{8}\left(\sqrt{3}+2\right)$$

d) Find the probability associated with a measurement that gives zero for z- component of the angular momentum. **Ans** $\left(\sqrt{\frac{1}{8}}\right)^2 = \frac{1}{8}$

e) Calculate $\langle\varphi|\hat{L}_z|\psi\rangle$ and $\langle\varphi|\hat{L}_-|\psi\rangle$ where

$$|\varphi\rangle = \sqrt{\frac{8}{15}}Y_{2,1}+\sqrt{\frac{4}{15}}Y_{1,0}+\sqrt{\frac{3}{15}}Y_{2,-1}$$

Ans:

$$\langle\varphi|\hat{L}_z|\psi\rangle = 0,$$

$$\langle\varphi|\hat{L}_-|\psi\rangle = \left\langle\sqrt{\frac{8}{15}}Y_{2,1}+\sqrt{\frac{4}{15}}Y_{1,0}+\sqrt{\frac{3}{15}}Y_{2,-1}\middle|\hat{L}_-\middle|\sqrt{\frac{3}{8}}Y_{1,1}+\sqrt{\frac{1}{8}}Y_{1,0}+\sqrt{\frac{1}{2}}Y_{1,-1}\right\rangle$$

$$=\left\langle\sqrt{\frac{4}{15}}Y_{1,0}\middle|\sqrt{2}\sqrt{\frac{3}{8}}Y_{1,0}\right\rangle = \sqrt{\frac{4}{15}}\sqrt{2}\sqrt{\frac{3}{8}} = \frac{\hbar}{\sqrt{5}}$$

E- Solve only two problems and show your work

2- An electron in a hydrogen atom is in a state given by the wave function:

$$\psi(\boldsymbol{r}) = \frac{1}{\sqrt{3\pi}}re^{-r} \quad .$$

d- Find the value of ℓ for this state. Justify your answer.

e- If $\psi(\boldsymbol{r})$ is expanded in hydrogen eigenstates as:

$$\psi(\boldsymbol{r}) = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell}C_{n\ell m}\psi_{n\ell m}(\boldsymbol{r})$$

for which values of *n*, ℓ and *m* will be $C_{n\ell m}$ non-zero.

f- What is the probability that a measurement of the energy of this electron will give the value -13.6 eV ?

Answer:

Instructor Prof. Dr. I. Nasser

a- No angular dependence, so $\ell = 0$.

b- For the expression::

$$\psi(\mathbf{r}) = \sum_{n=1}^{\infty}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell} C_{n\ell m}\psi_{n\ell m}(\mathbf{r})$$

For the values, $n = 1, \quad \ell = 0, \quad m = 0 \; C_{n\ell m}$ will be non-zero.

c-

$$C_{100} = \int \psi\psi_o r^2 \sin\theta d\,\theta d\,\varphi dr = 4\pi \underbrace{\int_0^{\infty}\left(\frac{e^{-r}}{\sqrt{\pi}}\right)\left(\frac{re^{-r}}{\sqrt{3\pi}}\right) r^2 dr}_{\sqrt{3}/8\pi} = \frac{\sqrt{3}}{2}$$

$$\therefore P(-13.6) = |C_{100}|^2 = \frac{3}{4}$$

3- An electron in the Coulomb field of a proton is in a state described by the following spatial wave function

$$\psi(r) = A\left[3|1,0,0\rangle + 2|2,1,1\rangle - |2,1,0\rangle + \sqrt{10}\,|3,1,-1\rangle\right]$$

f- Find the value of *A*.
g- What is the expectation value of the energy?
h- What is the expectation value of the L^2?
i- What is the expectation value of the L_z?
j- What is the expectation value of the L_+?

Answer:

a) Find the value of *A*. $A = \dfrac{1}{2\sqrt{6}}$.

$$P_1 + P_2 + P_3 + P_4 = A^2\left[a_1^2 + a_2^2 + a_3^2 + a_4^2\right] = A^2\left[3^2 + 2^2 + (-1)^2 + (\sqrt{10})^2\right]$$

$$= A^2(24) = 1$$

$$\Rightarrow A = 1/2\sqrt{6}$$

b- What is the expectation value of the energy?

n	P_n	E_n
1	$\left(\frac{3}{\sqrt{24}}\right)^2 = \frac{9}{24}$	-13.6 eV

2	$\left(\frac{2}{\sqrt{24}}\right)^2+\left(\frac{-1}{\sqrt{24}}\right)^2=\frac{5}{24}$	$-\frac{13.6}{4}$ eV
3	$\left(\frac{\sqrt{10}}{\sqrt{24}}\right)^2=\frac{10}{24}$	$-\frac{13.6}{9}$ eV

$$< E >=\left(\frac{9}{24}\right)(-13.6 \text{ eV})+\left(\frac{5}{24}\right)\left(-\frac{13.6}{4}\text{ eV}\right)+\left(\frac{10}{24}\right)\left(-\frac{13.6}{9}\text{ eV}\right)=-6.44 \text{ eV}.$$

c- What is the expectation value of the L^2? $\hat{L}^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle$

l	P_l	L^2
0	$\left(\frac{3}{\sqrt{24}}\right)^2=\frac{9}{24}$	$0\,\hbar^2$
1	$\left(\frac{2}{\sqrt{24}}\right)^2+\left(\frac{-1}{\sqrt{24}}\right)^2+\left(\frac{\sqrt{10}}{\sqrt{24}}\right)^2=\frac{15}{24}$	$2\,\hbar^2$

$$< L^2 >=\left(\frac{10}{24}\right)(0\,\hbar^2)+\left(\frac{15}{24}\right)(2\,\hbar^2)=1.25\,\hbar^2$$

d- What is the expectation value of the L_z? $\boxed{\hat{L}_z|l,m\rangle = m\hbar|l,m\rangle}$

m	P_m	$\hat{L}_z$
0	$\left(\frac{3}{\sqrt{24}}\right)^2+\left(-\frac{1}{\sqrt{24}}\right)^2=\frac{10}{24}$	$0\,\hbar$
1	$\left(\frac{2}{\sqrt{24}}\right)^2=\frac{4}{24}$	$\hbar$
-1	$\left(\frac{\sqrt{10}}{\sqrt{24}}\right)^2=\frac{10}{24}$	$-\hbar$

$$< \hat{L}_z >=\left(\frac{10}{24}\right)(0\,\hbar)+\left(\frac{4}{24}\right)(\ \hbar)+\left(\frac{10}{24}\right)(\ -\hbar)=-0.25\,\hbar$$

e- $\hat{L}_+|\psi(r)\rangle = A\left[0+0-\underbrace{C_{+11}}_{\sqrt{2}}|2,1,1\rangle + \sqrt{10}\underbrace{C_{+10}}_{\sqrt{2}}|3,1,0\rangle\right]$

$$\langle\psi|\hat{L}_+|\psi\rangle = A^2\left[-2\sqrt{2}\langle 2,1,1|2,1,1\rangle\right] = -\frac{\sqrt{2}}{12}\hbar$$

3- A particle in the state:

$$\psi(x,y,z) = \sqrt{\frac{3}{8\pi}}\left(\frac{z-iy}{r}\right)$$

a- Write the above expression in terms of spherical harmonics.
b- What is the expectation value of the L^2?
c- Suppose L_z is measured. What are the possible results?
d- What is the probability of obtaining each result?
e- What is the expectation value of the L_z?
f- Find $\langle \hat{L}_- \rangle$.

Answer:

a) use $z = r\cos\theta$, $x = r\sin\theta\cos\varphi$ $y = r\sin\theta\sin\varphi$, then

$$\sqrt{\frac{3}{8\pi}}\left(\frac{x \pm iy}{r}\right) = \sin\theta\cos\varphi \pm i\,\sin\theta\sin\varphi = \sin\theta e^{\pm i\varphi} = \mp\sqrt{\frac{8\pi}{3}}Y_{1,\pm1}$$

$$\psi(x,y,z) = \sqrt{\frac{3}{8\pi}}\left(\frac{z-iy}{r}\right) = \sqrt{\frac{3}{8\pi}}\cos\theta - \sqrt{\frac{3}{8\pi}}\sin\theta\left(\underbrace{2i\,\sin\varphi}_{e^{i\varphi}-e^{-i\varphi}}\right)$$

$$= \frac{1}{\sqrt{2}}Y_{1,0} + \frac{1}{2}Y_{1,1} + \frac{1}{2}Y_{1,-1}$$

Another method

Instructor Prof. Dr. I. Nasser

$$\psi(x,y,z)=\sqrt{\frac{3}{8\pi}}\left(\frac{z-iy}{r}\right)=\sqrt{\frac{3}{8\pi}}\frac{z}{r}-\sqrt{\frac{3}{8\pi}}\frac{iy}{r}$$
$$=\frac{1}{\sqrt{2}}Y_{1,0}-\sqrt{\frac{3}{8\pi}}\frac{iy}{r}+\frac{1}{2}\sqrt{\frac{3}{8\pi}}\frac{x}{r}-\frac{1}{2}\sqrt{\frac{3}{8\pi}}\frac{x}{r}$$
$$=\frac{1}{\sqrt{2}}Y_{1,0}-\frac{1}{2}\sqrt{\frac{3}{8\pi}}\frac{x+iy}{r}+\frac{1}{2}\sqrt{\frac{3}{8\pi}}\frac{x-iy}{r}$$
$$=\frac{1}{\sqrt{2}}Y_{1,0}+\frac{1}{2}Y_{1,1}+\frac{1}{2}Y_{1,-1}$$

In ket notation:

$$|\psi\rangle=\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{2}|1\rangle+\frac{1}{2}|-1\rangle$$

b)..For each term $\ell=1$, and so measurement of L^2 is certain to give $\ell(\ell+1)\hbar^2=2\hbar^2$. The total angular momentum of the particle is $\sqrt{2}\hbar$.

c- the values are: 1, 0, -1

d- What is the expectation value of the L^2?

l	P_l	L^2
1	$\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{-1}{2}\right)^2+\left(\frac{1}{2}\right)^2=1$	$2\hbar^2$

$$<L^2>=\left(\frac{1}{2}\right)\left(2\hbar^2\right)+\left(\frac{1}{4}\right)\left(2\hbar^2\right)+\left(\frac{1}{4}\right)\left(2\hbar^2\right)=2\hbar^2$$

e-

m	P_m	$\hat{L}_z$
0	$\left(\frac{1}{\sqrt{2}}\right)^2=\frac{1}{2}$	$0\hbar$
1	$\left(-\frac{1}{2}\right)^2=\frac{1}{4}$	$\hbar$
-1	$\left(\frac{1}{2}\right)^2=\frac{1}{4}$	$-\hbar$

$$< \hat{L}_z >= \left(\frac{1}{2}\right)(0\,\hbar) + \left(\frac{1}{4}\right)(\ \hbar) + \left(\frac{1}{4}\right)(\ -\hbar) = 0\,\hbar$$

f-

$$\left\langle \hat{L}_- \right\rangle = \langle \psi | \hat{L}_- | \psi \rangle = \langle \psi | \left[\frac{1}{\sqrt{2}} \hbar \sqrt{2} \, |-1\rangle - \frac{1}{2} \hbar \sqrt{2} \, |0\rangle \right]$$

$$= +\frac{\hbar}{2} + \frac{\hbar}{2} = -\hbar$$