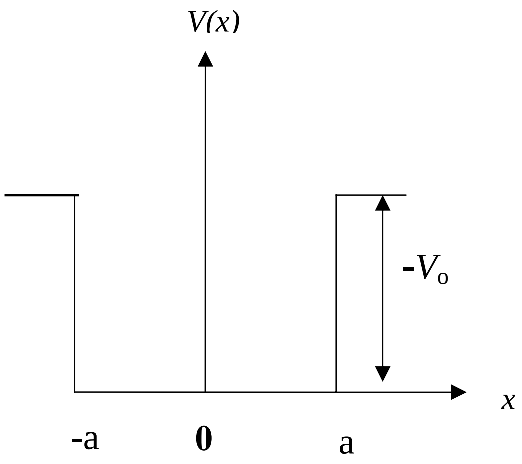
3- جسيم يتحرك داخل صندوق فى مجال جهد على الصورة:

$$V(x)=\begin{cases}0 & x<-a\\ -V_o & -a\le x\le a\\ 0 & x>a\end{cases}$$



Where the constant $V_o > 0$.

Case 1 $E > 0$,:

Define $\alpha^2 = \dfrac{2mE}{\hbar^2}$, and $\beta^2 = \dfrac{2m(E+V_o)}{\hbar^2}$, then the Schrödinger equations and its corresponding solutions are:

(1) if $x < -a$: $\psi_1'' + \alpha^2\psi_1 = 0 \;\Rightarrow\; \psi_1 = A_1 e^{i\alpha x} + B_1 e^{-i\alpha x}$

(2) if $-a \le x \le a$: $\psi_2'' + \beta^2\psi_2 = 0 \;\Rightarrow\; \psi_2 = A_2 e^{i\beta x} + B_2 e^{-i\beta x}$

(3) if $x > -a$: $\psi_3'' + \alpha^2\psi_3 = 0 \;\Rightarrow\; \psi_3 = A_3 e^{i\alpha x} + B_3 e^{-i\alpha x}$

Take $A_1 = 1$ and $B_3 = 0$ for no reflection in the right region.

Matching $\psi(x)$ and $\dfrac{d\psi(x)}{dx}$ at the boundaries a and –a, we can have:

$$e^{-i\alpha a} + B_1 e^{i\alpha a} = A_2 e^{-i\beta a} + B_2 e^{i\beta a}$$

$$i\alpha\left(e^{-i\alpha a} - B_1 e^{i\alpha a}\right) = i\beta\left(A_2 e^{-i\beta a} - B_2 e^{i\beta a}\right)$$

$$A_2 e^{i\beta a} + B_2 e^{-i\beta a} = A_3 e^{i\alpha a}$$

$$i\beta\left(A_2 e^{i\beta a} - B_2 e^{-i\beta a}\right) = i\alpha A_3 e^{i\alpha a}$$

which gives:

$$B_1 = ie^{-2i\alpha a}\,\frac{\left(\beta^2-\alpha^2\right)\sin\gamma}{2\alpha\beta\cos\gamma - i\left(\beta^2+\alpha^2\right)\sin\gamma}, \qquad \gamma = 2\beta a$$

$$A_3 = e^{-2i\alpha a}\,\frac{2\alpha\beta}{2\alpha\beta\cos\gamma - i\left(\beta^2+\alpha^2\right)\sin\gamma}$$

Comments:

$$1- \quad R = B_2^2 \geq 0, \quad R = 0 \text{ if } E \gg V_o,$$

$$2- \quad T + R = 1$$

$$3- \quad R = 0 \text{ if } \sin\gamma = 0$$

$$\Rightarrow 2\beta a = n\pi, \qquad n = 1,2,3,\cdots$$

Condition #3 implies

$$2\frac{2m(E+V_o)}{\hbar^2}a^2 = n^2\pi^2$$

Thus whenever the incident energy is give as

$$E = -V_o + \frac{n^2\pi^2\hbar^2}{8ma^2}$$

there is only transmission. This is called transmission resonance, and it was experimentally observed by Ramsauer and Townsend when they scattered low energy electron off noble atoms. Relate this resonance to de Broglie wave length. $\left(E \approx 0.1\,\text{eV}\right)$ off noble atoms.

Case II $E < V_o, \Rightarrow E = -\left|E\right|$

The Schrödinger equations and its corresponding solutions are:

(1) if $\quad x < -a: \quad \psi_1'' - \alpha^2\psi_1 = 0 \quad \Rightarrow \quad \psi_1 = A_1 e^{\alpha x} + B_1 e^{-\alpha x}$

(2) if $\quad -a \leq x \leq a: \quad \psi_2'' + \beta^2\psi_2 = 0 \quad \Rightarrow \quad \psi_2 = A_2\cos(\beta x) + B_2\sin(\beta x)$

(3) if $\quad x > -a: \quad \psi_3'' - \alpha^2\psi_3 = 0 \quad \Rightarrow \quad \psi_3 = A_3 e^{\alpha x} + B_3 e^{-\alpha x}$

In the region we used $\cos(\beta x)$ and $\sin(\beta x)$ rather than using $\mathrm{e}^{i\beta a}$ and $\mathrm{e}^{-i\beta a}$ because the outside solutions are real. Boundary at ∞ and $-\infty$ requires the coefficients A_3 and B_1 are zeros.

Matching the wave function and its derivatives at a and $-a$ gives

Where $\alpha^2 = -\frac{2m|E|}{\hbar^2}$, $\beta^2 = \frac{2m(V_o + |E|)}{\hbar^2}$. Matching $\psi(x)$ and $\frac{d\psi(x)}{dx}$ at the boundaries a and –a, we can have:

$$A_1 e^{-\alpha a} = A_2 \cos(\beta a) - B_2 \sin(\beta a)$$
$$\alpha A_1 e^{-\alpha a} = \beta(A_2 \sin(\beta a) + B_2 \cos(\beta a))$$
$$B_3 e^{-\alpha a} = A_2 \cos(\beta a) + B_2 \sin(\beta a)$$
$$-\alpha B_3 e^{-\alpha a} = -\beta(A_2 \sin(\beta a) - B_2 \cos(\beta a))$$

By adding and subtracting of these equations, we get a more lucid form of the system of equations, which is easy to solve:

$$(A_1 + B_3) e^{-\alpha a} = 2A_2 \cos(\beta a)$$
$$\alpha(A_1 + B_3) e^{-\alpha a} = 2\beta A_2 \sin(\beta a)$$
$$(A_1 - B_3) e^{-\alpha a} = -2B_2 \sin(\beta a)$$
$$\alpha(A_1 - B_3) e^{-\alpha a} = 2\beta B_2 \cos(\beta a)$$

Assuming that

$(A_1 + B_3) \neq 0$ and $A_2 \neq 0$, (or divided) the first two equations yield

$$\beta \tan(\beta a) = \alpha$$

Inserting this in one of the last two equations gives

$$A_1 = B_3; \qquad B_2 = 0$$

Hence, as a results, we have a symmetric solution with $\psi(x) = \psi(-x)$. We then speak of positive parity.

Almost identical calculations lead for

$(A_1 - B_3) \neq 0$ and for $B_2 \neq 0$ to

$$\beta \cot(\beta a) = -\alpha$$

and

$$A_1 = -B_3; \qquad A_2 = 0$$

We thus obtained an antisymmetric solution with $\psi(x) = \psi(x)$ corresponding to a negative parity.

Qualitative solution of the eigenvalue problem. The equations connecting κ and k, which we have already obtained, are conditions for the energy eigenvalue. Using the short forms

$$\xi = ka \ , \quad \eta = \kappa a \ , \tag{10}$$

we get from the definition (2)

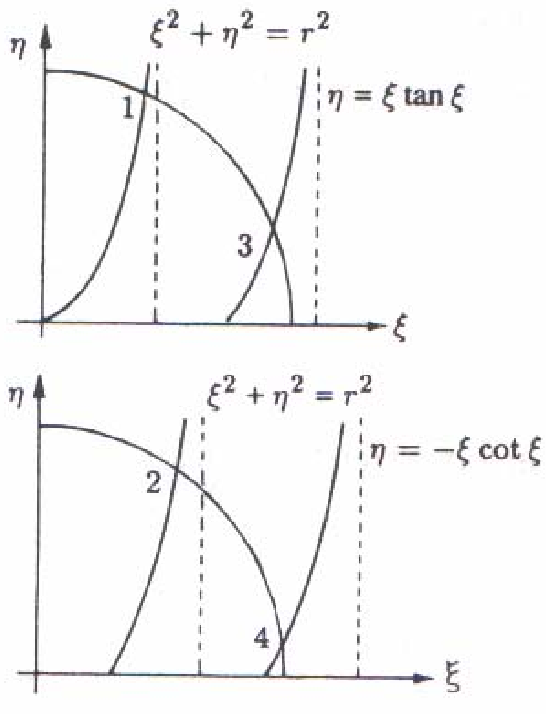
$$\xi^2 + \eta^2 = \frac{2mV_0a^2}{\hbar^2} = r^2 \ . \tag{11}$$

On the other hand, using (7) and (9) we get the equations

$$\eta = \xi \tan(\xi) \ , \quad \eta = -\xi \cot(\xi) \ .$$

Therefore the desired energy values can be obtained by constructing the intersection of those two curves with the circle defined by (11), within the (ξ, η) plane (see next figure).

At least one solution exists for arbitrary values of the parameter V_0, in the case of positive parity, because the tan function intersects the origin. For negative parity, the radius of the circle needs to be larger than a minimum value so that the two curves can intersect. The potential must have a certain depth in connection with a given size a and a given mass m, to permit a solution with negative



The intersections of these curves determine the energy eigenvalues parity. The number of energy levels increases with V_0, a and mass m. For the case $mVa^2 \to \infty$, the intersections are found at

$$\begin{aligned} \tan(ka) = \infty \quad &\text{corresponding to} \quad ka = \frac{2n-1}{2}\pi \ , \\ -\cot(ka) = \infty \quad &\text{corresponding to} \quad ka = n\pi \ , \\ &n = 1, 2, 3, \ldots \end{aligned} \tag{12}$$

or, combined:

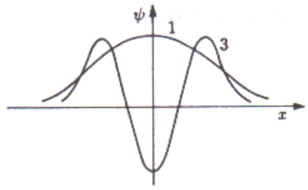
$$k(2a) = n\pi \ . \tag{13}$$

For the energy spectrum this means that

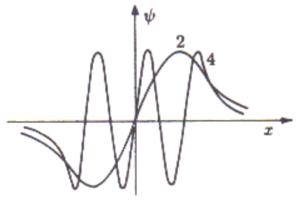
$$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2a}\right)^2 - V_0 \ . \tag{14}$$

On enlarging the potential well and/or the particle's mass m, the difference between two neighbouring energy eigenvalues will decrease. The lowermost state ($n = 1$) is not located at $-V_0$, but a little higher. This difference is called the *zero-point energy*. We will come back to it later when discussing the harmonic oscillator (see Chap. 7).

(e) The shape of the wave function is shown for the discussed solutions in the two figures.



Wave functions with positive parity; they are symmetric relative to the origin



Wave functions with negative parity; they are antisymmetric relative to the origin

Sins β and α are functions of the energy E, the last equation imposes restrictions on the values of energy E that permit a solution for A, B, C and D; in other words, energy quantization. Moreover, there are two types of solution, one obtain when

$\beta\tan(\beta a)=\alpha$ even solution

The other when

$\beta\cot(\beta a)=-\alpha$ odd solution .

Let's study them sequentially.

First

$\beta\tan(\beta a)=\alpha$

We have a set of four simultaneous linear homogeneous equations for four unknowns and the condition for a solution to exist is that the determinant of the equations vanish:

$$\begin{vmatrix} \cos(\beta a) & -\sin(\beta a) & -\mathrm{e}^{-\alpha a} & 0 \\ \beta\sin(\beta a) & \beta\cos(\beta a) & -\beta\,\mathrm{e}^{-\alpha a} & 0 \\ \cos(\beta a) & \sin(\beta a) & 0 & -\mathrm{e}^{-\alpha a} \\ -\beta\sin(\beta a) & \beta\cos(\beta a) & 0 & \beta\,\mathrm{e}^{-\alpha a} \end{vmatrix}=0$$

This leads to the equation

$$\left(\tan(\beta a)-\frac{\alpha}{\beta}\right)\left(\tan(\beta a)+\frac{\alpha}{\beta}\right)=0$$