KING FAHD UNIVERSITY of PETROLIUM and MINERALS
Physics Department
Quantum Mechanics I (Phys-401)
Spring 2008

Instructor: Prof. Dr. Ibraheem Nasser

Issued: 21-3-2009	HW#3	Due date: 28-3-2008

1- For a general 2x2 unitary matrix $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\det(U) = 1$, show that $a^* = d, b = -c^*$, and that $|a|^2 + |b|^2 = 1$. Show that such matrix has only two independent components.

2- The Hamiltonian for a two-state system is given by:

$$H = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}$$

A basis for this system is: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Find the eigenvalues and eigenvectors of H, and express the eigenvectors on terms of $\{|0\rangle, |1\rangle\}$

3- For the two dimensional rotation matrix

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

find the eigenvalues, the eigenvectors, and a unitary matrix that diagonalizes *R*.

4- For the Harmonic oscillator in one dimension, use the definitions

$$\hat{a} \equiv \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + i\hat{p}_x\right), \quad \hat{a}^\dagger \equiv \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} - i\hat{p}_x\right)$$

to prove the following:

a) $\left[\hat{a}, \hat{a}^\dagger\right] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$

b) $\hat{H} = \hbar\omega\left(\hat{a}\,\hat{a}^\dagger - \frac{1}{2}\right)$

c) $\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\,\hat{a}^\dagger\right); \quad \left[\hat{a}, H\right] = \hbar\omega\hat{a}; \quad \left[\hat{a}^\dagger, H\right] = -\hbar\omega\hat{a}^\dagger$

5- Using Pauli's matrices, If you define the following operators:

$$\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\,\sigma_y\right)$$

Prove that:

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

$$\sigma_+ |1\rangle = 0,$$

$$\sigma_+ |0\rangle = |1\rangle,$$

$$\sigma_- |1\rangle = |0\rangle,$$

$$\sigma_- |0\rangle = 0$$

6- If the orbital angular momentum operators are defined by:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

Prove the following commutations:

$$\left[\hat{L}_x, \hat{L}_y\right] = \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x = i\hbar\hat{L}_z$$

$$\left[\hat{x}, \hat{L}_x\right] = 0, \ \left[\hat{x}, \hat{L}_y\right] = i\hbar\hat{z}, \ \left[\hat{x}, \hat{L}_z\right] = -i\hbar\hat{y}$$

7- Show that: $e^{i\theta\sigma_x} = I\cos\theta + i\,\sigma_x \sin\theta$

8- Consider an operator $\hat{O} = -i\dfrac{d}{d\varphi}$ where φ is the azimuthal angle in spherical coordinate.

a. Find the eigenfunctions $f(\varphi)$ and eigenvalues λ subject to the constraint that $f(0) = f(2\pi) = \dfrac{1}{\sqrt{2\pi}}$ and that λ must be positive.

b. Consider another operator $\hat{\varphi}$ that acts as the position operator in Cartesian coordinates, i.e. $\hat{\varphi}|\psi\rangle = \varphi|\psi\rangle$. Find $\left[\hat{O}, \hat{\varphi}\right]$.

c. Is $\hat{O}$ Hermitian?

9- Put the operator (Hadamard gate):

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in outer product notation using the basis $\{|0\rangle, |1\rangle\}$. Calculate $H|0\rangle$ and $H|1\rangle$.

1- For $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\det(U) = 1 \quad \Rightarrow \quad ad - bc = 1$$

$$\Rightarrow \quad bc = ad - 1 \qquad (1)$$

$$UU^{\dagger} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & ac^* + bd^* \\ ca^* + db^* & |c|^2 + |d|^2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

So,

$$|a|^2 + |b|^2 = 1 \qquad (2)$$

$$ac^* + bd^* = 0 \qquad (3)$$

$$ca^* + db^* = 0 \qquad (4)$$

$$|c|^2 + |d|^2 = 1 \qquad (5)$$

Solving and rearranging equations (1) and (3) gives $a = d^*$. Same for the others will give $b = -c^*$. Then,

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

.

2- The Hamiltonian for a two-state system is given by:

$$H = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}$$

The eigenvalues are $\lambda_{\pm} = \omega_1 \pm \omega_2$ and eigenvectors of H,

	Eigenvalue	**Eigenstates**
$H = \begin{pmatrix} \omega_1 & \omega_2 \\ \omega_2 & \omega_1 \end{pmatrix}$	$\omega_1 + \omega_2$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \frac{1}{\sqrt{2}}\{\lvert 0\rangle + \lvert 1\rangle\}$
	$\omega_1 - \omega_2$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \frac{1}{\sqrt{2}}\{\lvert 0\rangle - \lvert 1\rangle\}$

3-

	Eigenvalue	Eigenstates
$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$	$e^{i\theta}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$
	$e^{-i\theta}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

4- See text, pages: 228 and 229

5- Easy to do
For example:

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$

$$\sigma_+ \left|1\right\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\sigma_+ \left|0\right\rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

6- See text, pages: 184, 270 and 271

7- See text, page 417

8-a- see text page 288,

$$\hat{O}f(\varphi) = -i\frac{d}{d\varphi}f(\varphi) = \lambda f(\varphi) \quad \Rightarrow \quad f(\varphi) = \frac{1}{\sqrt{2\pi}}e^{i\lambda\varphi}, \qquad \lambda = 1, 2, 3, \cdots$$

b- $\left[\hat{O}, \hat{\varphi}\right] = -i$

c- $\hat{O}$ is hermitian.

9- $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left\{|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right\}$

$$H\,|0\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle + |1\rangle\right\}$$

$$H\,|1\rangle = \frac{1}{\sqrt{2}}\left\{|0\rangle - |1\rangle\right\}$$