KING FAHD UNIVERSITY of PETROLIUM and MINERALS
Physics Department
Quantum Mechanics I (Phys-401)
Spring 2008
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Issued: 16-3-2009	HW#2	Due date: 21-3-2008

1- Write down the function $\psi(x)=\sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}$ in Dirac's notation.

2- If the initial wave function of a harmonic oscillator is given by

$$\psi(x,0)=\sqrt{\frac{1}{3}}\psi_0(x)+\sqrt{\frac{2}{3}}\psi_2(x)$$

where $\psi_0(x)$ and $\psi_2(x)$ are the eigen function for the ground state and the second excited states, respectively.

a) Calculate the wave function at any time.

b) Calculate the expectation value of x and E in the state $\psi(x,0)$.

3- Find the probability of finding the ground state harmonic oscillator outside the classical limits. Use $\psi_0(x)=\left(\frac{1}{x_o\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}x^2/x_o^2}$

4- Calculate the following brackets:

$\left[\hat{x},\hat{p}_x\right]$ ، $\left[\hat{x},\hat{p}_x^2\right]$ ، $\left[\hat{x}^2,\hat{p}_x\right]$, $\left[\hat{p}_x,\hat{V}\right]$ where $V(x)=a_0+a_1x+a_2x^2+\cdots$.

5- Prove the following:

$a)$ $\left[\left[\hat{x},\hat{p}_x^2\right],\hat{x}\right]=2\hbar^2$ $\qquad$ $b)$ $\hat{B}^{-1}\left[\hat{A},\hat{B}\right]\hat{B}^{-1}=-\left[\hat{A},\hat{B}^{-1}\right]$

$c)$ $[\hat{x}^3,\hat{p}_x]=3i\hbar\hat{x}^2$ $\qquad$ $d)$ $[\hat{H},\hat{x}]=[\frac{\hat{p}^2}{2m}+V(x),\hat{x}]=\frac{-2i\hbar}{2m}\hat{p}$

In the following problem, it is recommended to also use any software packages (Mathematica, Maple, Matlab, etc…)

6- Using the normalized, linear harmonic oscillator wave functions perform the necessary integrations to verify the following:

a) That ψ_o and ψ_1 and are normalized.

b) That ψ_o and ψ_1 and are orthogonal.

c) That $\langle \psi_o | x^2 | \psi_o \rangle = \frac{1}{2}$.

d) That $\langle \psi_1 | x^2 | \psi_1 \rangle = \langle \psi_1 | p_x^2 | \psi_1 \rangle = \frac{3}{2}$.

e) That $\langle \psi_0 | x | \psi_1 \rangle = \langle \psi_1 | x | \psi_0 \rangle = \frac{\sqrt{2}}{2}$.

f) Carry out the necessary differentiations and integrations to find the following expectation values for the first excited state of the harmonic oscillator: $\langle x \rangle$, $\langle x^2 \rangle$, $\langle p_x \rangle$, $\langle p_x^2 \rangle$.

g) Using the results of (f), find $\langle T \rangle, \langle V \rangle$, and $\langle E \rangle$ for the same state.

h) Calculate the minimum uncertainty product $\Delta x . \Delta p_x$ for this state.

[Use: $\hat{H} = \frac{p_x^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$, $\psi_n(x) = \left(\frac{\sqrt{\alpha}}{2^n n! \sqrt{\pi}} \right)^{1/2} e^{-\alpha x^2/2} H_n(\sqrt{\alpha} x)$]

1- Use the notation:

$$\psi(x) = \langle x|\psi\rangle = \sum_{n=1}^{\infty}\langle x|n\rangle\langle n|\psi\rangle = \sum_{n=1}^{\infty}\langle n|\psi\rangle\langle x|n\rangle,$$

Then $c_n = \langle n|\psi\rangle, \quad \langle x|n\rangle = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$

1- a- Remmember $E_n = (n+\frac{1}{2})\hbar\omega$

$$\begin{aligned}\psi(x,t) &= \sqrt{\frac{1}{3}}\psi_0(x)e^{-iE_o t/\hbar} + \sqrt{\frac{2}{3}}\psi_2(x)e^{-iE_2 t/\hbar}\\ &= \sqrt{\frac{1}{3}}\psi_0(x)e^{-i\omega t/2} + \sqrt{\frac{2}{3}}\psi_2(x)e^{-i\,5\omega t/2}\end{aligned}$$

b-

$$\begin{aligned}&\langle x\rangle = 0,\\ &\langle E\rangle = \frac{1}{3}E_o + \frac{2}{3}E_2 = \frac{1}{3}(\frac{1}{2}\hbar\omega) + \frac{2}{3}(\frac{5}{2}\hbar\omega) = \frac{11}{6}\hbar\omega\end{aligned}$$

3- The turning points are defined by E = U, i.e. $\frac{1}{2}\hbar\omega = \frac{1}{2}m\omega^2 x_o^2 \Rightarrow x_o = \pm\sqrt{\frac{\hbar}{m\omega}}$. The probability of finding the oscillator within the range of turning points is:

$$\int_{-x_o}^{x_o}|\psi_0(x)|^2 dx = \frac{1}{x_o\sqrt{\pi}}\int_{-x_o}^{x_o} e^{-x^2/x_o^2}dx = \frac{2}{\sqrt{\pi}}\int_0^1 e^{-y^2}dy = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\text{Erf}[1] = 0.843$$

Hence the probability of finding the ground state harmonic oscillator outside the classical limits (classical forbidden zone) $= 1 - 0.843 = 0.157$

4-

$$\begin{aligned}\left[\hat{x},\hat{D}_x\right]f(x) &= x\frac{d}{dx}\left[f(x)\right] - \frac{d}{dx}\left[xf(x)\right]\\ &= x\frac{d}{dx}f(x) - \left(x\frac{d}{dx}f(x) + f(x)\frac{d}{dx}x\right)\\ &= -f(x)\\ \Rightarrow &\quad \boxed{\left[\hat{x},\hat{D}_x\right] = \underline{-1}}\end{aligned}$$

$$\left[\hat{x}^2,\hat{D}_x\right]f(x)=x^2\frac{d}{dx}\left[f(x)\right]-\frac{d}{dx}\left[x^2f(x)\right]$$
$$=x^2\frac{d}{dx}f(x)-\left(x^2\frac{d}{dx}f(x)+f(x)\frac{d}{dx}x^2\right)$$
$$=-2xf(x).$$
$$\Rightarrow \boxed{\left[\hat{x}^2,\hat{D}_x\right]=\underline{-2x}}$$

$$\left[\hat{x},\hat{p}_x\right]=\left[\hat{x},\frac{\hbar}{i}\frac{\partial}{\partial x}\right]=\frac{\hbar}{i}\left[\hat{x},\frac{\partial}{\partial x}\right]=-\frac{\hbar}{i}\left[\frac{\partial}{\partial x},\hat{x}\right]=-\frac{\hbar}{i}=i\hbar$$

$$\left[\hat{x},\hat{p}_x^2\right]=\left[\hat{x},\hat{p}_x\right]\hat{p}_x+\hat{p}_x\left[\hat{x},\hat{p}_x\right]=i\hbar\frac{\hbar}{i}\frac{\partial}{\partial x}+\frac{\hbar}{i}\frac{\partial}{\partial x}.i\hbar=2\hbar^2\frac{\partial}{\partial x}=2i\hbar\hat{p}_x$$

$$\left[\hat{x}^2,\hat{p}_x\right]=\left[\hat{x}\hat{x},\hat{p}_x\right]=\hat{x}\left[\hat{x},\hat{p}_x\right]+\left[\hat{x},\hat{p}_x\right]\hat{x}=2i\hbar x$$

$$\left[\hat{p}_x,\hat{x}^2\right]=\left[\hat{p}_x,\hat{x}\hat{x}\right]=\left[\hat{p}_x,\hat{x}\right]\hat{x}+\hat{x}\left[\hat{p}_x,\hat{x}\right]=-i\hbar(2x)=-i\hbar\frac{d}{dx}(x^2)$$

$$\left[\hat{p}_x,\hat{x}^3\right]=\left[\hat{p}_x,\hat{x}^2\hat{x}\right]=\left[\hat{p}_x,\hat{x}\right]\hat{x}^2+\hat{x}\left[\hat{p}_x,\hat{x}^2\right]=-i\hbar(3x^2)=-i\hbar\frac{d}{dx}(x^3)$$

By inspection:

$$\left[\hat{p}_x,\hat{x}^n\right]=-i\hbar\frac{d}{dx}(x^n)$$

Then,

$$\left[\hat{p}_x,\hat{V}\right]=-i\hbar\left\{a_1+a_2\frac{dx^2}{dx}+a_3\frac{dx^3}{dx}+\cdots\right\}=-i\hbar\frac{d}{dx}V(x)$$

Using MATHEMATICA and the following wave function:

$$\psi_n(x)=\left(\frac{\sqrt{\alpha}}{2^n n!\sqrt{\pi}}\right)^{1/2}e^{-\alpha x^2/2}H_n(\sqrt{\alpha}x)$$

You can check the following results:

$$\hat{x}_{mn}=\langle\psi_m(x)|x|\psi_n(x)\rangle=\int_{-\infty}^{\infty}\psi_m(x)x\psi_n(x)dx=\begin{cases}0, & \text{if } m\neq n\pm1\\ \frac{1}{\alpha}\sqrt{\frac{n+1}{2}}, & \text{if } m=n+1\\ \frac{1}{\alpha}\sqrt{\frac{n}{2}}, & \text{if } m=n-1\end{cases}$$

$$\hat{x}^2_{nn} = \langle \psi_n(x) | x^2 | \psi_n(x) \rangle = \int_{-\infty}^{\infty} \psi_n(x) x^2 \psi_n(x) dx = \frac{2n+1}{2\alpha},$$

$$\left\langle \hat{p}_x \right\rangle = \langle \psi_n(x) | -i\hbar \frac{\partial}{\partial x} | \psi_n(x) \rangle = -i\hbar \int_{-\infty}^{\infty} \psi_n(x) \left(\frac{\partial}{\partial x} \psi_n(x) \right) dx = 0,$$

$$\left\langle \hat{p}_x^2 \right\rangle = \langle \psi_n(x) | -\hbar^2 \frac{\partial^2}{\partial x^2} | \psi_n(x) \rangle = -\hbar^2 \int_{-\infty}^{\infty} \psi_n(x) \left(\frac{\partial^2}{\partial x^2} \psi_n(x) \right) dx = \hbar^2 \alpha \left(\frac{2n+1}{2} \right),$$

$$\langle T \rangle = \frac{1}{2m} \left\langle \hat{p}_x^2 \right\rangle = \frac{\omega\hbar}{2} (n + \frac{1}{2}),$$

$$\langle V \rangle = \frac{m\omega^2}{2} \left\langle \hat{x}^2 \right\rangle = \frac{\omega\hbar}{2} (n + \frac{1}{2}),$$

$$\langle E \rangle = \langle T \rangle + \langle V \rangle = (n + \frac{1}{2}) \hbar\omega$$