KING FAHD UNIVERSITY of PETROLIUM and MINERALS
Physics Department
Quantum Mechanics I (Phys-401)
Spring 2008

Issued: 7-3-2009	Due date: 15-3-2009

1- Calculate the normalization constant A in the wave-function:

$$\psi = A\, x\, e^{-ar}, \qquad -\infty \le x \le \infty .$$

Hint: use the standard integrals: $\int_0^\infty x^n e^{-ax} dx = \frac{n!}{a^{n+1}}$; $\int_0^\pi \cos^n\theta \sin\theta d\theta = \frac{\{1+(-1)^n\}}{n+1}$;

and the spherical coordinates, in which $d\tau = r^2 \sin\theta d\theta d\varphi dr$, $x = r\sin\theta\cos\varphi$.

2- Assume the wave function of a particle is

$$\psi(x) = A\frac{e^{ip_o x}}{\sqrt{x^2+a^2}}$$

Here a and p_o are real constants and A is a normalization constant.

a. Find A so that $\psi(x)$ is normalized.
b. If the position of the particle is measured, what is the probability of finding the particle between $-\frac{a}{\sqrt{3}}$ and $\frac{a}{\sqrt{3}}$?
c. Calculate the mean value of the momentum of the particle.

3- Find the eigenfunctions and energy spectrum of a particle in the potential well given by:

$$V(x) = \begin{cases} 0, & |x| < L/2 \\ \infty, & |x| > L/2 \end{cases}$$

4- Consider the ground state of the particle in Problem 3.

i. Sketch the well and the wavefunction.
ii. Sketch the well and the probability density
iii. Show that the average position of the particle is x=0
iv. Show that the average momentum of the particle is zero.
v. Show that the normalization constant is $\sqrt{2/L}$.
vi. Show that $\Delta x \Delta p = 0.18\pi\hbar = 0.57\hbar$

1- Ans:

$$\int \psi^2 d\tau = A^2 \int x^2 e^{-2ar} d\tau = N^2 \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, r^4 e^{-2ar} \sin^3\theta \cos^2\varphi$$

$$= A^2 \int_0^\infty r^4 e^{-2ar} dr \int_0^\pi (1-\cos^2\theta)\sin\theta d\theta \int_0^{2\pi} \cos^2\varphi d\varphi$$

$$= A^2 \left\{\frac{4!}{2^5 a^5}\right\}\left\{2-\frac{2}{3}\right\}\{\pi\} = A^2 \frac{\pi}{a^5}$$

then $\int \psi^2 d\tau = 1 \Rightarrow A = \sqrt{\frac{a^5}{\pi}}$

2- (a)

$$\langle \psi|\psi \rangle = 1, \quad \int_{-\infty}^{+\infty} \psi^*(x)\psi(x)dx = 1, \quad N^2 \int_{-\infty}^{\infty} \frac{e^{-i\frac{p_0 x}{\hbar}}}{\sqrt{x^2+a^2}} \frac{e^{i\frac{p_0 x}{\hbar}}}{\sqrt{x^2+a^2}} dx =$$

$$N^2 \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} dx = 1, \quad \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)_{-\infty}^{\infty} = \frac{1}{a}\left[\frac{\pi}{2} - \frac{-\pi}{2}\right] = \frac{\pi}{a}.$$

$$N^2 = \frac{a}{\pi}, \quad N = \sqrt{\frac{a}{\pi}}$$

(b) Let P denote the probability of finding the particle between $-\frac{a}{\sqrt{3}}$ and $+\frac{a}{\sqrt{3}}$.

$$P = \int_{-\frac{a}{\sqrt{3}}}^{\frac{a}{\sqrt{3}}} |\psi(x)|^2 dx = N^2 \int_{-\frac{a}{\sqrt{3}}}^{\frac{a}{\sqrt{3}}} \frac{1}{x^2+a^2} dx = \frac{1}{\pi}\tan^{-1}\left(\frac{x}{a}\right)_{-\frac{a}{\sqrt{3}}}^{\frac{a}{\sqrt{3}}}$$

$$= \frac{1}{\pi}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{6} + \frac{\pi}{6}\right] = \frac{1}{3}.$$

(c)

Let <P> denote the mean value of the momentum of the particle.

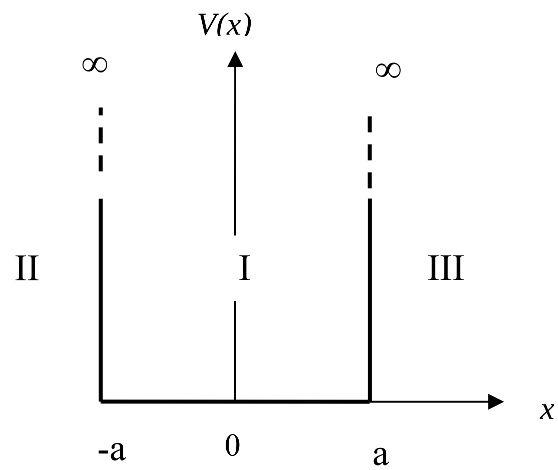
$$\langle P \rangle = \int_{-\infty}^{\infty} \psi^*(x) \frac{\hbar\partial}{i\partial x} \psi(x) dx$$

$$= N^2 \int_{-\infty}^{\infty} \frac{e^{-i\frac{p_0 x}{\hbar}}}{\sqrt{x^2+a^2}} \frac{\hbar}{i}\left[\frac{ip_0}{\hbar}\frac{e^{i\frac{p_0 x}{\hbar}}}{\sqrt{x^2+a^2}} - \frac{e^{i\frac{p_0 x}{\hbar}}}{(x^2+a^2)^{\frac{3}{2}}} x\right] dx$$

$$= N^2 p_0 \underbrace{\int_{-\infty}^{\infty} \frac{1}{x^2+a^2} dx}_{\frac{1}{N^2}} - N^2 \underbrace{\int_{-\infty}^{\infty} \frac{x}{(x^2+a^2)^2} dx}_{0,\ \text{odd function}} = p_0.$$

3- Find the eigenfunctions and energy spectrum of a particle in the potential well given by:

$$V(x) = \begin{cases} 0, & |x| < a \\ \infty, & |x| > a \end{cases}$$



$$-\frac{\hbar^2}{2m}\frac{d^2\psi_I}{dx^2} = E\,\psi_I \Rightarrow$$

$$\frac{d^2\psi_I}{dx^2} = -\alpha^2\psi_I\,, \qquad \alpha^2 = \frac{2mE}{\hbar^2}$$

$$\psi_I(x) = A\sin(\alpha x) + B\cos(\alpha x) \qquad (1)$$

Boundary conditions at a and -a implies

$$\psi_I(a) = 0 \quad \Rightarrow \quad A\cos(\alpha a) + B\sin(\alpha a) = 0, \qquad (2)$$

$$\psi_I(-a) = 0 \quad \Rightarrow \quad A\cos(\alpha a) - B\sin(\alpha a) = 0 \qquad (3)$$

Adding equations (2) and (3), we have

$$2A\cos(\alpha a) = 0 \quad \Rightarrow \quad A = 0 \text{ or } \cos(\alpha a) = 0$$

Subtracting equations (2) and (3), we have

$$2B\sin(\alpha a) = 0 \quad \Rightarrow \quad B = 0 \text{ or } \sin(\alpha a) = 0$$

Now we don't want both *A* and *B* to be zero, since this would give the physically uninteresting solution $\psi_I(x)$. Also, we can't make both $\sin(\alpha a)$ and $\cos(\alpha a)$ to be zero, for a given value of α and *E*. There are two possible classes of solutions:

(i) $A = 0,\ B \neq 0 \quad \Rightarrow \quad \sin(\alpha a) = 0$

(ii) $B = 0,\ A \neq 0 \quad \Rightarrow \quad \cos(\alpha a) = 0$

For the first class

$$\sin(\alpha a) = 0 \quad \Rightarrow \quad \alpha a = \pi, 2\pi, 3\pi, \cdots = \frac{n\pi}{2}, n = \text{ even integer}$$

For the second class

$$\cos(\alpha a) = 0 \quad \Rightarrow \quad \alpha a = \frac{\pi}{2}, 3\frac{\pi}{2}, 2\pi, \cdots = \frac{n\pi}{2}, n = \text{ odd integer}$$

Thus

$$\psi_I(x) = \begin{cases} A \sin(\frac{n\pi}{2a} x) & n \text{ is even} \\ \\ B \cos(\frac{n\pi}{2a} x) & n \text{ is odd} \end{cases}$$

$$\alpha^2 = \frac{2mE}{\hbar^2}, \qquad \Rightarrow \qquad E = \frac{\alpha^2 \hbar^2}{2m} = \frac{\pi^2 \hbar^2}{2ma^2} n^2$$

And

$$A = B = \sqrt{\frac{1}{a}}$$

In both cases. To transfer it our problem, use $a = L/2$

Thus there is infinite sequence of discrete energy levels that corresponding to all positive integer values of the quantum number *n*.

MATHEMATICA program has been used for problem 4

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<<Graphics`Legend`

(* Problem 4  *)

Ψo[a_, x_, n_] = Sqrt[1/a] Sin[(n π)/(2 a) x]
```

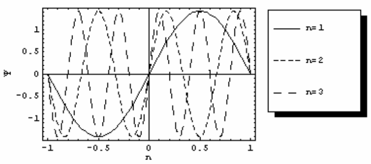
$\sqrt{\frac{1}{a}}\ \mathrm{Sin}\left[\frac{n \pi x}{2 a}\right]$

```
a = 0.5

0.5

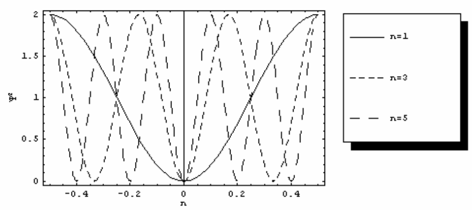
y1 = Ψo[a, x, 1]; y3 = Ψo[a, x, 3]; y5 = Ψo[a, x, 5] ;

Plot[{y1, y3, y5}, {x, -2 a , 2 a}, Frame -> True, PlotStyle -> {GrayLevel[0],
   Dashing[{0.02}], Dashing[{0.04}]}, PlotLegend -> {"n=1", "n=2", "n=3", "n=5"},
 LegendPosition -> {1.1, -0.2},
 FrameLabel -> {n, Ψ}]
```



- Graphics -

```
Plot[{y1^2, y3^2, y5^2}, {x, -a , a}, Frame -> True, PlotStyle -> {GrayLevel[0],
   Dashing[{0.02}], Dashing[{0.04}]}, PlotLegend -> {"n=1", "n=3", "n=5"}, LegendPosition -> {1.1, -0.2},
 FrameLabel -> {n, Ψ^2}]
```



- Graphics -

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<<Graphics`Legend`

(* Problem 4  *)

Ψe[a_, x_, n_] = Sqrt[1/a] Cos[(n π)/(2 a) x]

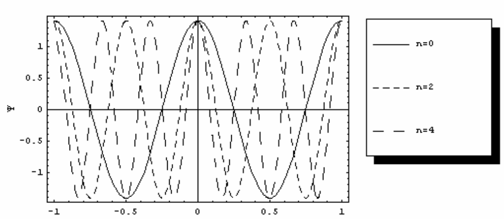
1.41421 Cos[3.14159 n x]

a = 0.5

0.5

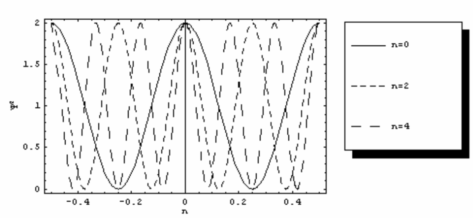
y2e = Ψe[a, x, 2]; y4e = Ψe[a, x, 4]; y6e = Ψe[a, x, 6] ;

Plot[{y2e, y4e, y6e}, {x, -2 a , 2 a}, Frame -> True, PlotStyle -> {GrayLevel[0],
   Dashing[{0.02}], Dashing[{0.04}]}, PlotLegend -> {"n=0", "n=2", "n=4"}, LegendPosition -> {1.1, -0.2},
 FrameLabel -> {n, Ψ}]
```



- Graphics -

```
Plot[{y2e^2, y4e^2, y6e^2}, {x, -a , a}, Frame -> True, PlotStyle -> {GrayLevel[0],
   Dashing[{0.02}], Dashing[{0.04}]}, PlotLegend -> {"n=0", "n=2", "n=4"}, LegendPosition -> {1.1, -0.2},
 FrameLabel -> {n, Ψ^2}]
```



- Graphics -

```
<< Graphics`Legend`
a=1
```

$$\Psi e[a\_, x\_, n\_] = \sqrt{\frac{1}{a}}\ \mathrm{Cos}\left[\frac{n\,\pi}{2\,a}\,x\right]$$

$$\mathrm{Cos}\left[\frac{n\,\pi\,x}{2}\right]$$

$$aa = \mathrm{Integrate}\left[\ \Psi e[a, x, 1]^2, \{x, -a, a\}, \mathrm{Assumptions} \to a > 0\right]$$

$$1$$

$$xa = \mathrm{Integrate}\left[x\,\Psi e[a, x, 1]^2, \{x, -a, a\}, \mathrm{Assumptions} \to a > 0\right]$$

$$0$$

$$x2a = \int_{-a}^{a} x^2\ \Psi e[a, x, 1]^2\, dx\ \text{// Simplify}$$

$$\frac{1}{3} - \frac{2}{\pi^2}$$

$$pa = -i\,h\,\mathrm{Integrate}[\Psi e[a, x, 1]\ \partial_x(\Psi e[a, x, 1]), \{x, -a, a\}, \mathrm{Assumptio}$$

$$0$$

$$p2a = (-i\,h)^2\ \int_{-a}^{a} \Psi e[a, x, 1]\ \partial_x(\partial_x(\Psi e[a, x, 1]))\, dx$$

$$\frac{h^2\,\pi^2}{4}$$

$$\Delta x = \sqrt{x2a - xa^2}$$

$$\sqrt{\frac{1}{3} - \frac{2}{\pi^2}}$$

$$\Delta p = \sqrt{p2a - pa^2}$$

$$\frac{\sqrt{h^2}\ \pi}{2}$$

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Simplify[Δx Δp] // N
```

$$0.567862\,\sqrt{h^2}$$