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KING FAHD UNIVERSITY of PETROLIUM and MINERALS
Physics Department
Quantum Mechanics I (Phys-401)
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A- Give a brief but reasoned answer to each of the following:

a) The matrices representing two observables $\hat{A}$ and $\hat{B}$ are:

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Can $\hat{A}$ and $\hat{B}$ be measured simultaneously? No, because $\left[\hat{A}, \hat{B}\right] \neq 0$

b) Show that the commutator $\left[\hat{A}, \hat{B}\hat{C}\right]$ can be expressed as:

$$\left[\hat{A}, \hat{B}\hat{C}\right] = \hat{B}\left[\hat{A}, \hat{C}\right] + \left[\hat{A}, \hat{B}\right]\hat{C}$$

Ans:

$$\left[\hat{A}, \hat{B}\hat{C}\right] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A}$$

$$= \hat{B}\left[\hat{A}, \hat{C}\right] + \left[\hat{A}, \hat{B}\right]\hat{C}$$

c) A particle of mass m in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$ (where $\omega \equiv \sqrt{k/m}$ is in the (normalized) state:

$$|\psi, t=0\rangle = \frac{1}{\sqrt{6}}\left(|0\rangle + i\,|1\rangle - 2|2\rangle\right)$$

where $|n\rangle$ are the energy eigenfunctions of the harmonic oscillator corresponding to the energies $E_n = (n + \frac{1}{2})\hbar\omega$. What is the average value of the energy in this state.

Ans:

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle$$

$$\hat{H}|\psi\rangle = \frac{1}{\sqrt{6}}\left(\hat{H}|0\rangle + i\hat{H}|1\rangle - 2\hat{H}|2\rangle\right)$$

$$= \frac{1}{\sqrt{6}}\left(E_o|0\rangle + iE_1|1\rangle - 2E_2|2\rangle\right) = \frac{1}{\sqrt{6}}\left(\frac{1}{2}|0\rangle + i\,\frac{3}{2}|1\rangle - 2\frac{5}{2}|2\rangle\right)\hbar\omega$$

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle = \left(\frac{1}{6}\times\frac{1}{2} + \frac{1}{6}\times\frac{3}{2} + \frac{4}{6}\times\frac{5}{2}\right)\hbar\omega = 2\hbar\omega$$

d) In a certain region of x the wave function is $|\psi\rangle = Be^{-ikx}$. What is the probability current $j(x)$ in this region?

Ans: $j(x) = \frac{\hbar}{2mi}\left[B^* e^{ikx}(-ikBe^{-ikx}) - (ikB^* e^{ikx})Be^{-ikx}\right] = -\frac{\hbar k}{m}|B|^2 = -v\,|B|^2$

e) Calculate the commutator $\left[\frac{d}{dx},(bx+c)\right]$, where b and c are constant, Ans: b

Ans: $\left[\frac{d}{dx},(bx+c)\right]=b\underbrace{\left[D_x,x\right]}_{=1}+\underbrace{\left[D_x,c\right]}_{=0}$

B- Show your work

1) A particle of mass m in a one-dimensional harmonic oscillator potential $V(x)=\frac{1}{2}kx^2=\frac{1}{2}m\omega^2x^2$ (where $\omega\equiv\sqrt{k/m}$ has the initial wavefunction

$$|\psi,t=0\rangle=N\left(\sqrt{2}|0\rangle-|3\rangle\right),$$

where $|n\rangle$ are the energy eigenfunctions of the harmonic oscillator corresponding to the energies $E_n=(n+\frac{1}{2})\hbar\omega$.

a) Find N such that $|\psi(t=0)\rangle$ is properly normalized. Ans: $N=1/\sqrt{3}$

b) What is the wavefunction at later time, $|\psi,t\rangle$?

Ans: $|\psi,t\rangle=\frac{1}{\sqrt{3}}\left(\sqrt{2}|0\rangle e^{-i\omega t/2}-|3\rangle e^{-i7\omega t/2}\right),\qquad E_n=(n+\frac{1}{2})\hbar\omega$

c) Compute the expectation value of the energy $\langle\psi,t|\hat{H}|\psi,t\rangle$ in terms of ω.

Ans:

$$\langle\psi,t|\hat{H}|\psi,t\rangle=\left(\frac{1}{\sqrt{3}}\right)^2\left(\left(\sqrt{2}\right)^2E_o+(-1)^2E_3\right)=\left(\frac{2}{3}E_o+\frac{1}{3}E_3\right)$$

$$=\left(\frac{2}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{7}{2}\right)=\frac{3}{2}\hbar\omega$$

d) If you make a measurement of the particle's energy, what possible values could you measure?

Ans: E_o and E_3

e) What is the most probable result for a measurement of the particle\s energy? E_o

f) What is the expectation vale for the particle's potential energy, $\langle\psi,t|\frac{1}{2}m\omega^2x^2|\psi,t\rangle$.

Ans: Use $\frac{1}{2}m\omega^2x^2=\frac{\hbar\omega}{4}\left(a+a^\dagger\right)^2=\frac{\hbar\omega}{4}\left(a^2+a^{\dagger 2}+1+2a^\dagger a\right)^2$. All terms are zero except $1+2a^\dagger a$. These two terms give

$$\langle\psi,t|\frac{1}{2}m\omega^2x^2|\psi,t\rangle=\frac{\hbar\omega}{4}|N|^2\left\{2\langle 0|\left(1+2a^\dagger a\right)|0\rangle+\langle 3|\left(1+2a^\dagger a\right)|3\rangle\right\}$$

$$=\frac{\hbar\omega}{12}\left\{2(0+1)+(6+1)\right\}=\frac{3}{4}\hbar\omega$$

2) Consider 2-by-2 Hermitian matrix in the form:

$$H = \begin{pmatrix} 1 & \dfrac{1-i}{\sqrt{2}} \\ h_{21} & 1 \end{pmatrix}$$

(a) What is the value of h_{21}? Ans: $\dfrac{1+i}{\sqrt{2}}$

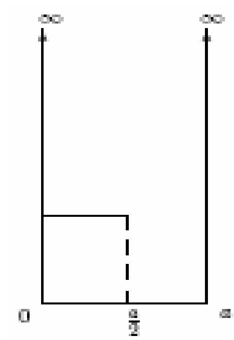
(b) Write down the characteristic polynomial of *H*. Ans: $P(\lambda) = (1-\lambda)^2 - 1$

(c) Find the eigenvalues and the normalized eigenvectors of *H*.

Eigenvalue	Normalized Eigenvector
2	$\frac{1}{2}\begin{pmatrix} 1-i \\ \sqrt{2} \end{pmatrix}$
0	$\frac{1}{2}\begin{pmatrix} i-1 \\ \sqrt{2} \end{pmatrix}$

3) A particle of mass m is moving in the infinite potential well shown in the figure has an initial wave function $\psi(x,0)$ given by:

$$\psi(x,0) = \begin{cases} A & \text{for} \quad 0 \le x < a/2 \\ 0 & \text{for} \quad a/2 \le x < a \end{cases}$$



where *A* is a real constant.

(a) Determine A by normalizing $\psi(x,0)$.

(b) Find $\langle x \rangle, \langle x^2 \rangle$, and $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

Ans:

(a) $$1 = |A|^2 \int_0^{a/2} dx = |A|^2 \left(\frac{a}{2}\right) \quad \Rightarrow \quad |A| = \sqrt{\frac{2}{a}},$$

(b)

$$\langle x \rangle = \frac{2}{a}\int_0^{a/2} x\, dx = \frac{2}{a}\frac{x^2}{2}\Big|_0^{a/2} = \frac{a}{4},$$

$$\langle x \rangle = \frac{2}{a}\int_0^{a/2} x^2\, dx = \frac{2}{a}\frac{x^3}{3}\Big|_0^{a/2} = \frac{a^2}{12},$$

$$\sigma_x = \sqrt{\frac{a^2}{12} - \frac{a^2}{16}} = \frac{a}{4\sqrt{3}}$$

4) The Hamiltonian *H* of two level system with energy eigenvalues E_1 and E_2 and corresponding eigenstates $|1\rangle$ and $|2\rangle$ has the matrix form::

$$H = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$$

(a) Find the energy eigenvalues E_1 and E_2 in terms of *E*.

(b) Calculate the orthonormal eigenvectors $|1\rangle$ and $|2\rangle$.

Eigenvalue	Normalized Eigenvector
E	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
-E	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) At time t = 0 the system is in the state $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, Express $|\psi(0)\rangle$ as a linear combination of the energy eigenvectors. Ans: $|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left\{|1\rangle + |2\rangle\right\}$;

(d) Express $|\psi(t)\rangle$ as a linear combination of the energy eigenvectors. Ans:

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left\{|1\rangle e^{-iEt/\hbar} + |2\rangle e^{iEt/\hbar}\right\}$$

(e) Calculate $\langle\psi(0)|\psi(t)\rangle$. Ans: $\langle\psi(0)|\psi(t)\rangle = \cos(Et/\hbar)$

5) Let $\left\{|0\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, |1\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ be an orthonormal two dimensional basis and let operators $\hat{A}$ and $\hat{B}$ are given by:

$$\hat{A} = |0\rangle\langle 0|, \qquad \hat{B} = |1\rangle\langle 1|$$

a- Find the matrix representation of $\hat{A}$ and $\hat{B}$. $\hat{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad \hat{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

b- Are these operators projection operators? Why?Ans:
Check with the conditions $P = P^2, \quad P = P^\dagger$

c- Let $|\psi\rangle = \{2i\,|1\rangle - 4|0\rangle\}$. Is $|\psi\rangle$ normalized? If not, normalize the state. $\langle\psi|\psi\rangle = 20$

d- Fin the representation of the normalized $|\psi\rangle$ as a column vector. Ans: $|\psi\rangle = \frac{1}{\sqrt{20}}\begin{pmatrix} 2i \\ -4 \end{pmatrix}$

e- Find the action of $\hat{A}$ and $\hat{B}$ on the normalized state $|\psi\rangle$, using both the outer product notation and

Ans:
$$\hat{A}\,|\psi\rangle = |0\rangle\langle 0|\psi\rangle = -\frac{2}{\sqrt{5}}|0\rangle,$$
$$\hat{B}\,|\psi\rangle = |1\rangle\langle 1|\psi\rangle = \frac{i}{\sqrt{5}}|1\rangle$$

f- by the action of the matrices representing these operators.

Ans:
$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}|\psi\rangle = \frac{1}{\sqrt{20}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2i \\ -4 \end{pmatrix} = -\frac{2}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{5}}|0\rangle$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}|\psi\rangle = \frac{1}{\sqrt{20}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 2i \\ -4 \end{pmatrix} = \frac{2i}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2i}{\sqrt{5}}|1\rangle$$