KING FAHD UNIVERSITY of PETROLIUM and MINERALS
Physics Department
Quantum Mechanics I (Phys-401)
T-082

Issued: 4-4-2009	HW#4	Due date: 11-4-2009

1) A particle of mass m in a one-dimensional harmonic oscillator potential $V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$ (where $\omega \equiv \sqrt{k/m}$ has the initial wavefunction

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right),$$

a) Find $|\psi(t)\rangle$

b) Define the expectation value by $\langle x(0)\rangle = \langle\psi(0)|x|\psi(0)\rangle$, calculate $\langle x(0)\rangle, \langle p(0)\rangle, \langle x(t)\rangle, \langle p(t)\rangle$

c) Find $\langle \dot{x}(t)\rangle, \langle \dot{p}(t)\rangle$ using Ehrenfest's theorem and solve for $\langle x(t)\rangle, \langle p(t)\rangle$ and compare to part b.

2) A particle in an infinite square well (of width *a and bounded at* $x = \pm 1$) has as its initial wave function an equal mixture of the first two stationary states:

$$\psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right]$$

(a) Normalize $\psi(x,0)$ to find *A*.

(b) Find $\psi(x,t)$ and $\psi(x,t)^2$. Express the latter in terms of sin and cos using $e^{i\theta} = \cos\theta + i\sin\theta$. Use $\theta = \hbar\pi^2/2ma^2$.

(c) Compute $\langle x\rangle$. Notice that it oscillates in time. What is the frequency of the oscillation? What is the amplitude?

(d) Compute $\langle \hat{p}\rangle$.

(e) Find the expectation value of the Hamiltonian operator, *H*, in terms of E_1 and E_2.

Use the wave functions:

$$\psi_1 = \left(\frac{2}{a}\right)^{\frac{1}{2}} \cos\left(\frac{\pi}{a}x\right),$$

$$\psi_2 = \left(\frac{2}{a}\right)^{\frac{1}{2}} \sin\left(\frac{2\pi}{a}x\right)$$

$$E_n = n^2\left(\frac{\hbar^2\pi^2}{2ma^2}\right) = n^2\hbar\omega, \qquad n = 1, 2, \cdots$$

3) With the help of problem 3.12, solve exercise 3.22.

1)At $t=0$ a particle starts out in the state $\left|\psi(0)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + \left|1\right\rangle\right)$,

a. Find $\left|\psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left(e^{-iE_o t/\hbar}\left|0\right\rangle + e^{-iE_1 t/\hbar}\left|1\right\rangle\right)$

$$\left\langle x\,(0)\right\rangle = \left\langle \psi(0)\right| x \left|\psi(0)\right\rangle = \sqrt{\frac{\hbar}{2m\omega}},$$

$$\left\langle p\,(0)\right\rangle = 0,$$

b. Find

$$\left\langle x\,(t\,)\right\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t\,),$$

$$\left\langle p\,(t\,)\right\rangle = -\sqrt{\frac{2m\omega}{\hbar}}\sin(\omega t\,)$$

c. Find

$$\left\langle \dot{x}(t)\right\rangle = \frac{\left\langle p(t)\right\rangle}{m},$$

$$\left\langle \dot{p}(t)\right\rangle = -m\omega^2\left\langle x(t)\right\rangle$$

Solve the coupled differential equation, one can find:

$$\left\langle x(t)\right\rangle = \left\langle x(0)\right\rangle\cos(\omega t) + \frac{\left\langle p(0)\right\rangle}{m}\sin(\omega t),$$

$$\left\langle p(t)\right\rangle = \left\langle p(0)\right\rangle\sin(\omega t) - m\omega^2\left\langle x(0)\right\rangle\cos(\omega t)$$

Find $\langle \dot{X}(t)\rangle$ and $\langle \dot{P}(t)\rangle$ using Ehrenfest's theorem and solve for $\langle X(t)\rangle$ and $\langle P(t)\rangle$ and compare to part 2

For $\langle \dot{X}(t)\rangle$,

$$\begin{aligned}\langle \dot{X}(t)\rangle &= \left(\frac{-i}{\hbar}\right)\langle [X,H]\rangle \\ &= \left(\frac{-i}{\hbar}\right)\left(\left\langle \left[X,\frac{P^2}{2m}\right]\right\rangle + \langle [X,V(x)]\rangle\right) \\ &= \left(\frac{-i}{4m^2\hbar}\right)\left(\langle P[X,P]\rangle + \langle [X,P]P\rangle\right) \\ &= \frac{\langle P(t)\rangle}{m}\end{aligned}$$

For $\langle \dot{P}(t)\rangle$

$$\begin{aligned}\langle \dot{P}(t)\rangle &= \left(\frac{-i}{\hbar}\right)\langle [P,H]\rangle \\ &= \left(\frac{-i}{\hbar}\right)\left(\left\langle \left[P,\frac{P^2}{2m}\right]\right\rangle + \langle [P,V(x)]\rangle\right) \\ &= -i\hbar\langle\frac{dV}{dx}\rangle = -m\omega^2\langle X(t)\rangle\end{aligned}$$

For this to be true, the time dependent $\langle X(t)\rangle$ has to contain the momentum term. With that some initi needs to be chosen. In this case, $t=0$. So

$$\langle X(t)\rangle = \langle X(t=0)\rangle\cos\omega t + \left\langle\frac{P(t=0)}{m}\right\rangle\sin\omega t$$

Likewise

$$\langle P(t)\rangle = \langle P(t=0)\rangle\sin\omega t - m\omega^2\langle X(t=0)\rangle\cos\omega t$$

2) A particle in an infinite square well (of width *a and bounded at* $x = \pm 1$) has as its initial wave function an equal mixture of the first two stationary states:

$$\psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right]$$

a) To find A, we will use the normalization condition $\int_{-a/2}^{a/2} |\psi(x,0)|^2 dx = 1$ to have

$$\begin{aligned}1 &= C^2 \int_{-a/2}^{a/2} \left(|\psi_1|^2 + |\psi_2|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1\right)dx \\ &= C^2(1+1+0+0) \\ &\Rightarrow C = 1/\sqrt{2}\end{aligned}$$

b) $|\psi(x,t)|^2$

$$\begin{aligned}\psi(x,t) = \psi(x,0)e^{-E_n t/\hbar} &= C\left[\psi_1(x)e^{-iE_1 t/\hbar} + \psi_2(x)e^{-iE_2 t/\hbar}\right] \\ &= C\left[\psi_1(x)e^{-E_1 t/\hbar} + \psi_2(x)e^{-E_2 t/\hbar}\right]\end{aligned}$$

And

$$\begin{aligned}|\psi(x,t)|^2 &= C^2\left[\psi_1(x)e^{-iE_1 t/\hbar} + \psi_2(x)e^{-iE_2 t/\hbar}\right]^2 \\ &= C^2\left(|\psi_1|^2 + |\psi_2|^2 + \psi_1\psi_2\, e^{i(E_1-E_2)t/\hbar} + \psi_2\psi_1\, e^{-i(E_1-E_2)t/\hbar}\right). \\ &= C^2\left(|\psi_1|^2 + |\psi_2|^2 + 2\psi_2\psi_1 \cos 3\omega t\right)\end{aligned}$$

Where we used $\left(E_1 - E_2\right) = 1^2\hbar\omega - 2^2\hbar\omega = 3\hbar\omega$

c) $\langle x \rangle$

$$\begin{aligned}\langle x \rangle &= \int_{-a/2}^{a/2} |\psi(x,t)|^2\, x\, dx = \int_{-a/2}^{a/2} C^2\left(|\psi_1|^2 + |\psi_2|^2 + 2\psi_2\psi_1\cos 3\omega t\right) x\, dx \\ &= C^2 \int_{-a/2}^{a/2} \left(\cancel{|\psi_1|^2} + \cancel{|\psi_2|^2} + 2\psi_2\psi_1 \cos 3\omega t\right) x\, dx \\ &= 2C^2 \cos 3\omega t \int_{-a/2}^{a/2} \left(\left(\frac{2}{a}\right)^{\frac{1}{2}} \cos\left(\frac{\pi}{a}x\right)\left(\frac{2}{a}\right)^{\frac{1}{2}} \sin\left(\frac{2\pi}{a}x\right)\right) x\, dx \\ &= \underbrace{\frac{16a}{9\pi^2}}_{\text{Amplitude}} \cos\left(\underbrace{3\omega}_{\text{Freqency}}\ t\right)\end{aligned}$$

d) $\langle \hat{p} \rangle$

$$\langle p\rangle=\int_{-a/2}^{a/2}\psi(x,t)^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x,t)x\ dx=-i\hbar C^2\int_{-a/2}^{a/2}\left(\psi_1^*+\psi_2^*\right)\frac{\partial}{\partial x}\left(\psi_1+\psi_2\right)dx$$

$$=-\frac{4}{3a}i\hbar\left(e^{i(E_2-E_1)t/\hbar}-e^{-i(E_2-E_1)t/\hbar}\right)$$

$$=\underbrace{\frac{4\hbar}{3a}}_{\text{Amplitude}}\sin\left(\underbrace{3\omega}_{\text{Freqency}}t\right)$$

e) $\langle H\rangle$

$$\langle H\rangle=\int_{-a/2}^{a/2}\psi(x,t)^*\left(H\right)\psi(x,t)x\ dx=C^2\int_{-a/2}^{a/2}\left(\psi_1^*+\psi_2^*\right)\left(H\psi_1+H\psi_2\right)dx$$

$$=C^2\int_{-a/2}^{a/2}\left(\psi_1^*+\psi_2^*\right)\left(E_1\psi_1+E_2\psi_2\right)dx=C^2\int_{-a/2}^{a/2}\left(E_1\left|\psi_1\right|^2+E_2\left|\psi_2\right|^2\right)dx$$

$$=\left(E_1+E_2\right)/2$$

Note that:

$$\int_{-a/2}^{a/2}\psi_1^*\psi_2\ dx=\int_{-a/2}^{a/2}\psi_2^*\psi_1\ dx=0$$