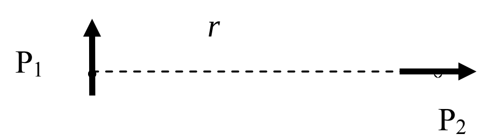
KING FAHD UNIVERSITY OF PETROLUIM AND MINERALS
DEPARTMENT OF PHYSICS

PHYS 305
ELECTRICITY AND MAGNETISM I
SPRING 2008
SECOND MAJOR (15/5/2008)
TIME (10:00 -> 12:00 A.M.)

Answer the following problems. (SHOW YOUR WORK)

1- In the following figure, P_1 and P_2 are perfect dipoles a distance "r" apart. What is the torque (magnitude and direction) on P_1 due to P_2?

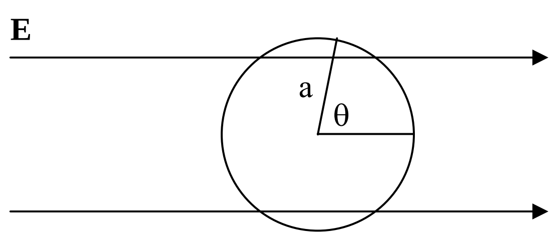


Answer: Taking $\vec{P}_2 = |\vec{P}_2|\,\hat{r}$, and $\vec{P}_1 = |\vec{P}_1|\,\hat{\theta}$, the electric field of P_2 at P_1 (where $\theta = \pi$) will be given by:

$$\vec{E}_2 = k\,\frac{P_2}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) = k\,\frac{P_2}{r^3}\left(2\hat{r}\right) \text{ (points to the right).}$$

Torque on P_1 is: $\vec{N}_1 = \vec{P}_1 \times \vec{E}_2 = k\,\frac{2P_1P_2}{r^3}$ (Points into the page).

2- A conducting sphere of radius "a" bearing total charge "Q" is placed in an initially uniform electric field $\vec{E} = E_o\,\hat{z}$. Find the potential at all points exterior to the sphere.



Answer: Start with the expression for the potential in the form:

$$V(r,\theta)=\frac{Q}{r}+\sum_{n=0}^{\infty}\left(A_n r^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Now, using the B. C, as $r\to\infty$ we wish the potential to behave as $-E_o z=-Er\cos\theta$ Thus $A_1=-E_o$ and all others A_n's are zero.

$$V(r,\theta)=\frac{Q}{r}-E_o r\cos\theta+\sum_{n=0}^{\infty}B_n r^{-n-1}P_n(\cos\theta)$$

Now, we have to make sure that this potential satisfy the B.C. at $r = a$,

$$V(a,\theta)=\frac{Q}{a}=\frac{Q}{a}-E_o a\cos\theta+\sum_{n=0}^{\infty}B_n a^{-n-1}P_n(\cos\theta)$$

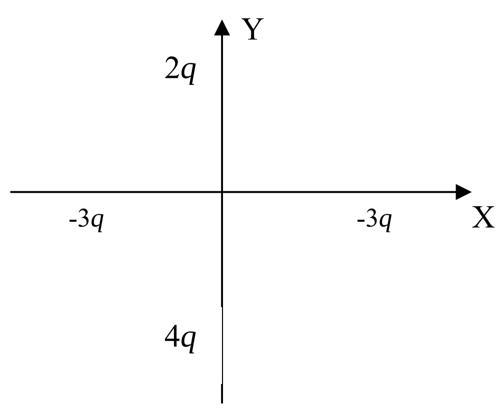
$$=-E_o a\cos\theta+\frac{B_o}{a}+\frac{B_1}{a^2}P_1(\cos\theta)+\frac{B_2}{a^3}P_2(\cos\theta)+\cdots$$

$$\Rightarrow B_o=0,\quad B_1=E_o a^3,\quad B_n=0\quad \text{for } n\neq 0,1.$$

Thus the final solution that satisfy all B. C. is

$$\boxed{V(r,\theta)=\left(\frac{a^3}{r^3}-1\right)E_o r\cos\theta+\frac{Q}{r}}.$$

3- Calculate the potential, then the electric field, at a distance *r* from the configuration in the figure. The charges are *at* equidistance *a* from the origin.



Ans:

$$V(r)=V_{monopole}+V_{dipole};$$

$$V_{monopole}=k\,\frac{Q}{r};$$

$$V_{dipole}=k\,\frac{\vec{p}.\hat{r}}{r^2}.$$

Where

$$Q = \sum_{i=1}^{4} q_i = 6 - 6 = 0;$$

$$\vec{p} = \sum_{i=1}^{5} q_i r_i = (-2q)(-a)\,\hat{\text{j}} - 4qa\,\hat{\text{j}} - 3q(a)\,\hat{\text{i}} + 3qa\,\hat{\text{i}} = 2qa\,\hat{\text{j}};$$

$$\vec{p}.\hat{\text{r}} = 2qa(\sin\theta\sin\varphi\,\hat{\text{r}} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\theta\,\hat{\varphi})\cdot\hat{\text{r}} = 2qa\sin\theta\sin\varphi.$$

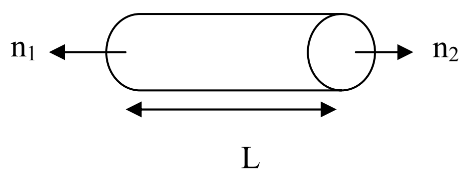
Then

$$V_{dipole} = k\,\frac{\vec{p}.\hat{r}}{r^2} = -k\,\frac{2qa\sin\theta\sin\varphi}{r^2}.$$

4- A thin dielectric rod of cross section A extends along the x-axis from x=0 to x=L. The polarization of the rod is along its length and is given by:

$$P = ax^2 + b$$

a- Find the volume polarization density and the surface polarization density on each end.
b- Calculate the total polarization charge.



$$\rho_b = -\nabla.\vec{P} = -\frac{d}{dx}P = -2ax\,,$$

$$\sigma_b(0) = \vec{P}.\hat{\text{n}}_1 = -\vec{P}.\hat{\text{r}}\Big|_{x=0} = -P\Big|_{x=0} = -(\cancel{ax^2} + b) = -b\,,$$

$$\sigma_b(\text{L}) = \vec{P}.\hat{\text{n}}_2 = \vec{P}.\hat{\text{r}}\Big|_{x=L} = P\Big|_{x=L} = -(ax^2 + b)\Big|_{x=L} = aL^2 + b\,,$$

Total polarization charge:

$Q_{total} = \sigma_b$ Area $+ \rho_b$ Volume

$$= \int \rho_b\,d\tau + \int \sigma_b\,dA = \int_0^L -2ax\;A\,dx + (AL^2 + b - b)$$

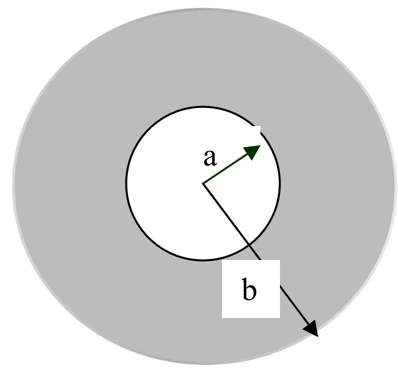
$$= \underbrace{-aL^2A}_{\text{from the volume}} \;\; \underbrace{+aL^2A}_{\text{from the area}} = 0$$

5- The following figure shows the cross section of a long conducting cylinder with inner radius a and outer radius b, $b > a$. The cylinder carries a current out of the page, and the current density in the cross section is given by $J = cr^2$, where c is a constant.

a- Find the current enclosed by the Amperian loop of radius r, $a<r<b$.

b- What is the magnetic field (magnitude and direction) at point r, $a < r < b$, from the center of the cylinder?

Answer:



a-

$$i_{enc} = \int JdA = \int_a^r cr^2 (2\pi r dr) = 2\pi c \int_a^r r^3 dr = \frac{\pi c(r^4 - a^4)}{2}$$

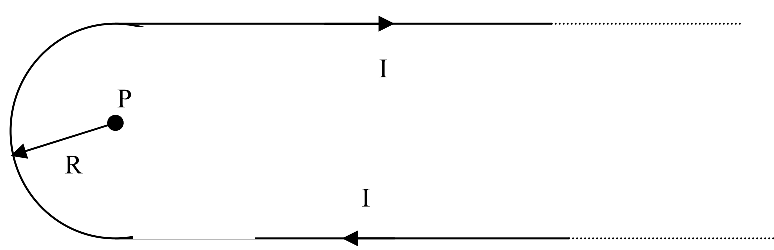
b-

$$\oint \vec{B}.d\vec{s} = \mu_o i_{enc} \Rightarrow B(2\pi r) = -\frac{\mu_o \pi c(r^4 - a^4)}{2}$$

$$\Rightarrow B = -\frac{\mu_o c(r^4 - a^4)}{4r}$$

and will be in the counterclockwise direction.

6- Find the magnetic field (magnitude and direction) at point P for the steady current configuration shown in the following figure.



Answer:

The two half-lines are the same as one infinite line: $\frac{\mu_o I}{2\pi R}$; the half-circle contributes $\frac{\mu_o I}{4R}$. So,

$B = \frac{\mu_o I}{4R}(1+\frac{2}{\pi})$ (into the page).

7- In spherical coordinate, if $\vec{A}(r,\theta,\phi) = r\sin\theta\,\hat{\phi}$, find the magnetic field.

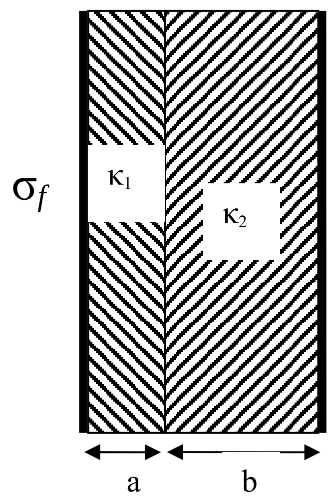
Answer:

$$\begin{aligned}\vec{B} &= \nabla\times\vec{A} = \frac{1}{r sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\hat{r} - \frac{1}{r}\frac{\partial}{\partial\theta}(rA_\phi)\hat{\theta}\right] \\ &= 2\left[\cos\theta\,\hat{r} - sin\theta\,\hat{\theta}\right]\end{aligned}$$

8- A parallel plate capacitor is filled with two dielectric slabs of thickness a and b. The dielectric constants of the slabs are κ_1 and κ_2 respectively. .

A- By what factor is the capacitance increased compare to the capacitance when no dielectric is present?.

B- Determine the bound surface charge density at the interface between the two dielectrics, if the free charge density on the capacitor plate is σ_f .



Answer: If the area of the plate is A and the separation between them is d, then

In vacuum:

$$C_o = \frac{Q}{V_o} = \frac{\sigma_f A}{E_o d} = \frac{\sigma_f A}{(\sigma_f/\varepsilon_o)d} = \varepsilon_o\frac{A}{d}$$

where we used $E_o = \frac{\sigma_f}{\varepsilon_o}$. Notice that, in the presence of the dielectric material we can have,

$E = \frac{E_o}{\kappa}$, $V = \frac{V_o}{\kappa}$, $C = \kappa C_o$. The capacitance with the dielectric is calculated as:

$$C = \frac{Q}{V} = \frac{\sigma_f A}{E_1 a + E_2 b} = \frac{\sigma_f A}{\left(\frac{\sigma_f}{\varepsilon_o\kappa_1}\right)a + \left(\frac{\sigma_f}{\varepsilon_o\kappa_2}\right)b} = \varepsilon_o A\frac{\kappa_1\kappa_2}{b\kappa_1 + a\kappa_2}$$

And

$$\frac{C}{C_o} = (a+b)\left(\frac{\kappa_1\kappa_2}{b\kappa_1 + a\kappa_2}\right) = d\left(\frac{\kappa_1\kappa_2}{b\kappa_1 + a\kappa_2}\right)$$

Another method:

For the series connection of two capacitors we have:

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{\left(\frac{\varepsilon_o A}{a}\kappa_1\right)\left(\frac{\varepsilon_o A}{b}\kappa_2\right)}{\left(\frac{\varepsilon_o A}{a}\kappa_1\right) + \left(\frac{\varepsilon_o A}{b}\kappa_2\right)} = \varepsilon_o A\left(\frac{\kappa_1\kappa_2}{b\kappa_1 + a\kappa_2}\right)$$

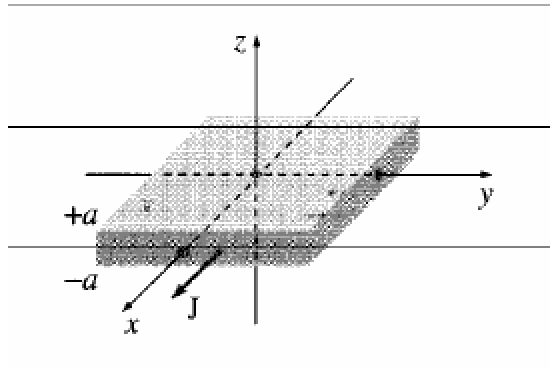
$$\therefore \frac{C}{C_o} = d\left(\frac{\kappa_1\kappa_2}{b\kappa_1 + a\kappa_2}\right) = (a+b)\left(\frac{\kappa_1\kappa_2}{b\kappa_1 + a\kappa_2}\right)$$

B- At the interface

$$\sigma_{b1} = P_1 = \sigma_f\,(1 - \frac{1}{\kappa_1})$$

$$\sigma_{b2} = P_2 = -\sigma_f\,(1 - \frac{1}{\kappa_2})$$

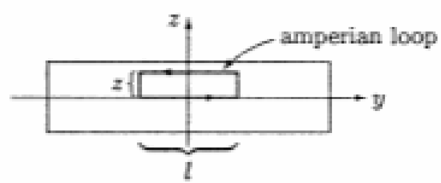
9- A thick slab extending from $z=-a$ to $z=+a$ carries a uniform volume current as shown in the figure. Find the magnetic field, as a function of z, both inside and outside the slab.



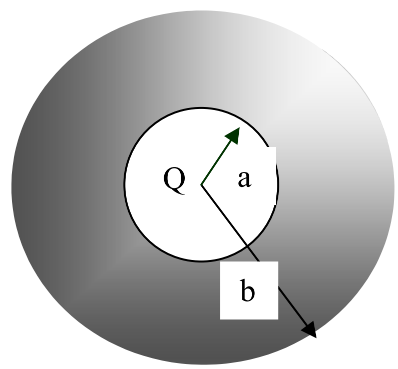
By the right-hand-rule, the field points in the $-\hat{\mathbf{y}}$ direction for $z>0$, and in the $+\hat{\mathbf{y}}$ direction for $z<0$. At $z=0$, $B=0$. Use the amperian loop shown:

$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{enc} = \mu_0 l z J \Rightarrow \boxed{\mathbf{B} = -\mu_0 J z\, \hat{\mathbf{y}}}$ $(-a<z<a)$. If $z>a$, $I_{enc} = \mu_0 l a J$,

so $\boxed{\mathbf{B} = \left\{ \begin{array}{ll} -\mu_0 J a\, \hat{\mathbf{y}}, & \text{for } z>+a; \\ +\mu_0 J a\, \hat{\mathbf{y}}, & \text{for } z>-a. \end{array} \right\}}$



10- A dielectric spherical shell (with inner radius a and outer radius b) has a dipole moment density (polarization) $\vec{P} = c\,\vec{r}$, where c is constant. A charge Q is placed at the center.



A- Find the electric field $\vec{E}$ in the three regions by first calculating $\vec{D}$.

B- Find the bound charge ρ_b in the dielectric and σ_b on the outer surface of the shell.

Answer:

A- Due to the spherical symmetry, we can use Gauss' law in the form:

$$\oint \vec{\nabla}.\vec{D}\, d\tau \equiv \oint \vec{D}.d\vec{a} = \oint \rho_f\, d\tau = Q$$

$$\Rightarrow \vec{D} = \frac{1}{4\pi}\frac{\mathrm{Q}}{r^2}\hat{\mathrm{r}}$$

$$\vec{E} = \frac{1}{\varepsilon_o}\left(\vec{D} - \vec{P}\right)$$

$$r < a: \quad \vec{P} = 0 \quad \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon_o} = k\,\frac{Q}{r^2}\hat{\mathrm{r}}$$

$$a < r < b: \; \vec{P} = c\vec{r} \quad \Rightarrow \vec{E} = k\,\frac{\mathrm{Q}}{r^2}\hat{\mathrm{r}} - \frac{c}{\varepsilon_o}\vec{r}$$

$$b < r: \quad \vec{P} = 0 \quad \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon_o} = k\,\frac{Q}{r^2}\hat{\mathrm{r}}$$

B-

$$\rho_b = -\nabla.\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 cr) = -3c$$

$$\sigma_b = \vec{P}.\hat{\mathrm{n}} = \begin{cases} -\vec{P}.\hat{\mathrm{r}}\Big|_{r=a} = -ca \\ \vec{P}.\hat{\mathrm{r}}\Big|_{r=b} = cb \end{cases}$$