KING FAHD UNIVERSITY OF PETROLUIM AND MINERALS DEPARTMENT OF PHYSICS <u>PHYS 305</u> <u>Electricity and Magnetism I</u> <u>Spring 2008</u> Second Major (15/5/2008) TIME (10:00 -> 12:00 A.M.)

Answer the following problems. (SHOW YOUR WORK)

1- In the following figure, P_1 and P_2 are perfect dipoles a distance "r" apart. What is the torque (magnitude and direction) on P_1 due to P_2 ?



Answer: Taking $\vec{P}_2 = |\vec{P}_2|\hat{r}$, and $\vec{P}_1 = |\vec{P}_1|\hat{\theta}$, the electric field of P_2 at P_1 (where $\theta = \pi$) will be given by:

$$\vec{E}_2 = k \frac{P_2}{r^3} \left(2\cos\theta \,\hat{\mathbf{r}} + \sin\theta \,\hat{\theta} \right) = k \frac{P_2}{r^3} \left(2\hat{r} \right)$$
 (points to the right).

Torque on P_1 is: $\vec{N}_1 = \vec{P}_1 \times \vec{E}_2 = k \frac{2P_1P_2}{r^3}$ (Points into the page).

2- A conducting sphere of radius "a" bearing total charge "Q" is placed in an initially uniform electric field $\vec{E} = E_o \hat{z}$. Find the potential at all points exterior to the sphere.



Answer: Start with the expression for the potential in the form:

$$V(r,\theta) = \frac{Q}{r} + \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta)$$

Now, using the B. C, as $r \to \infty$ we wish the potential to behave as $-E_o z = -Er \cos \theta$ Thus $A_1 = -E_o$ and all others A_n 's are zero.

$$V(r,\theta) = \frac{Q}{r} - E_o r \cos\theta + \sum_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos\theta)$$

Now, we have to make sure that this potential satisfy the B.C. at r = a,

$$V(a,\theta) = \frac{Q}{a} = \frac{Q}{a} - E_o a \cos\theta + \sum_{n=0}^{\infty} B_n a^{-n-1} P_n(\cos\theta)$$
$$= -E_o a \cos\theta + \frac{B_o}{a} + \frac{B_1}{a^2} P_1(\cos\theta) + \frac{B_2}{a^3} P_2(\cos\theta) + \cdots$$
$$\Rightarrow B_o = 0, \quad B_1 = E_o a^3, \quad B_n = 0 \quad \text{for } n \neq 0, 1.$$

Thus the final solution that satisfy all B. C. is

$$V(r,\theta) = \left(\frac{a^3}{r^3} - 1\right) E_o r \cos\theta + \frac{Q}{r}.$$

3- Calculate the potential, then the electric field, at a distance *r* from the configuration in the figure. The charges are *at* equidistance *a* from the origin.



Ans:

$$V(r) = V_{monopole} + V_{dipole};$$
$$V_{monopole} = k \frac{Q}{r};$$
$$V_{dipole} = k \frac{\vec{p} \cdot \hat{r}}{r^{2}}.$$

Where

$$Q = \sum_{i=1}^{4} q_i = 6 - 6 = 0;$$

$$\vec{p} = \sum_{i=1}^{5} q_i r_i = (-2q)(-a) \hat{j} - 4qa \hat{j} - 3q(a) \hat{i} + 3qa \hat{i} = 2qa \hat{j};$$

 $\vec{p}.\hat{\mathbf{r}} = 2qa(\sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\mathbf{\theta}} + \cos\theta\,\hat{\mathbf{\phi}})\cdot\hat{\mathbf{r}} = 2qa\sin\theta\sin\varphi.$

Then

$$V_{dipole} = k \frac{\vec{p}.\hat{r}}{r^2} = -k \frac{2qa\sin\theta\sin\phi}{r^2}.$$

4- A thin dielectric rod of cross section A extends along the x-axis from x=0 to x=L. The polarization of the rod is along its length and is given by:

$$P = ax^2 + b$$

- a- Find the volume polarization density and the surface polarization density on each end.
- b- Calculate the total polarization charge.



$$\begin{split} \rho_b &= -\nabla . \vec{P} = -\frac{d}{dx} P = -2ax ,\\ \sigma_b(0) &= \vec{P} . \hat{\mathbf{n}}_1 = -\vec{P} . \hat{\mathbf{r}}\Big|_{x=0} = -P \Big|_{x=0} = -(ax^2 + b) = -b ,\\ \sigma_b(\mathbf{L}) &= \vec{P} . \hat{\mathbf{n}}_2 = \vec{P} . \hat{\mathbf{r}}\Big|_{x=L} = P \Big|_{x=L} = -(ax^2 + b) \Big|_{x=L} = aL^2 + b , \end{split}$$

Total polarization charge: $Q_{total} = \sigma_b$ Area + ρ_b Volume

$$= \int \rho_b d\tau + \int \sigma_b dA = \int_0^L -2ax \ A \, dx + (AL^2 + b - b)$$
$$= \underbrace{-aL^2A}_{\text{from the volume}} \underbrace{+aL^2A}_{\text{from the area}} = 0$$

- 5- The following figure shows the cross section of a long conducting cylinder with inner radius *a* and outer radius *b*, b > a. The cylinder carries a current out of the page, and the current density in the cross section is given by $J = cr^2$, where *c* is a constant.
- a- Find the current enclosed by the Amperian loop of radius r, a < r < b.
- b- What is the magnetic field (magnitude and direction) at point r, a < r < b, from the center of the cylinder?

Answer:

a-

$$i_{enc} = \int J dA = \int_{a}^{r} cr^{2} (2\pi r dr) = 2\pi c \int_{a}^{r} r^{3} dr = \frac{\pi c (r^{4} - a^{4})}{2}$$

b-

$$\oint \vec{B}.d\vec{s} = \mu_o i_{enc} \Rightarrow B(2\pi r) = -\frac{\mu_o \pi c(r^4 - a^4)}{2}$$
$$\Rightarrow B = -\frac{\mu_o c(r^4 - a^4)}{4r}$$



and will be in the counterclockwise direction.

6- Find the magnetic field (magnitude and direction) at point P for the steady current configuration shown in the following figure.



Answer:

The two half-lines are the same as one infinite line: $\frac{\mu_o I}{2\pi R}$; the half-circle contributes $\frac{\mu_o I}{4R}$. So,

$$B = \frac{\mu_o I}{4R} (1 + \frac{2}{\pi}) \text{ (into the page).}$$

7- In spherical coordinate, if $\vec{A}(r,\theta,\phi) = r \sin \theta \hat{\phi}$, find the magnetic field.

Answer:

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (\sin \theta A_{\phi}) \hat{r} - \frac{1}{r} \frac{\partial}{\partial \theta} (r A_{\phi}) \hat{\theta} \right]$$
$$= 2 \left[\cos \theta \, \hat{r} - \sin \theta \, \hat{\theta} \right]$$

- 8- A parallel plate capacitor is filled with two dielectric slabs of thickness a and b. The dielectric constants of the slabs are κ_1 and κ_2 respectively.
- A- By what factor is the capacitance increased compare to the capacitance when no dielectric is present?.
- B- Determine the bound surface charge density at the interface between the two dielectrics, if the free charge density on the capacitor plate is σ_f .

Answer: If the area of the plate is A and the separation between them is d, then

In vacuum:

$$C_{o} = \frac{Q}{V_{o}} = \frac{\sigma_{f}A}{E_{o}d} = \frac{\sigma_{f}A}{\left(\sigma_{f}/\varepsilon_{o}\right)d} = \varepsilon_{o}\frac{A}{d}$$

where we used $E_o = \frac{\sigma_f}{\varepsilon_o}$. Notice that, in the presence of the dielectric material we can have,

 $E = \frac{E_o}{\kappa}$, $V = \frac{V_o}{\kappa}$, $C = \kappa C_o$. The capacitance with the dielectric is calculated as:

$$C = \frac{Q}{V} = \frac{\sigma_f A}{E_1 a + E_2 b} = \frac{\sigma_f A}{\left(\frac{\sigma_f}{\varepsilon_o \kappa_1}\right) a + \left(\frac{\sigma_f}{\varepsilon_o \kappa_2}\right) b} = \varepsilon_o A \frac{\kappa_1 \kappa_2}{b \kappa_1 + a \kappa_2}$$

And



$$\frac{C}{C_o} = (a+b) \left(\frac{\kappa_1 \kappa_2}{b \kappa_1 + a \kappa_2} \right) = d \left(\frac{\kappa_1 \kappa_2}{b \kappa_1 + a \kappa_2} \right)$$

Another method:

For the series connection of two capacitors we have:

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{\left(\frac{\varepsilon_o A}{a} \kappa_1\right) \left(\frac{\varepsilon_o A}{b} \kappa_2\right)}{\left(\frac{\varepsilon_o A}{a} \kappa_1\right) + \left(\frac{\varepsilon_o A}{b} \kappa_2\right)} = \varepsilon_o A \left(\frac{\kappa_1 \kappa_2}{b \kappa_1 + a \kappa_2}\right)$$
$$\therefore \frac{C}{C_o} = d \left(\frac{\kappa_1 \kappa_2}{b \kappa_1 + a \kappa_2}\right) = (a+b) \left(\frac{\kappa_1 \kappa_2}{b \kappa_1 + a \kappa_2}\right)$$

B- At the interface

$$\sigma_{b1} = P_1 = \sigma_f \left(1 - \frac{1}{\kappa_1}\right)$$
$$\sigma_{b2} = P_2 = -\sigma_f \left(1 - \frac{1}{\kappa_2}\right)$$

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9- A thick slab extending from z = -a to z = +a carries a uniform volume current as shown in the figure. Find the magnetic field, a function of z, both inside and outside the slab.



By the right-hand-rule, the field points in the $-\hat{\mathbf{y}}$ direction for z > 0, and in the $+\hat{\mathbf{y}}$ direction for z < 0. At z = 0, B = 0. Use the amperian loop shown: $\oint \mathbf{B} \cdot d\mathbf{l} = Bl = \mu_0 I_{enc} = \mu_0 lz J \Rightarrow \boxed{\mathbf{B} = -\mu_0 Jz \, \hat{\mathbf{y}}} (-a < z < a). \text{ If } z > a, I_{enc} = \mu_0 la J,$ so $\boxed{\mathbf{B} = \left\{ \begin{array}{c} -\mu_0 Ja \, \hat{\mathbf{y}}, & \text{for } z > +a; \\ +\mu_0 Ja \, \hat{\mathbf{y}}, & \text{for } z > -a. \end{array} \right\}}$ $\underbrace{z = 0, B = 0. \text{ Use the amperian loop for } y$

10- A dielectric spherical shell (with inner radius a and outer radius b) has a dipole moment density (polarization) $\vec{P} = c \vec{r}$, where c is constant. A charge Q is placed at the center.



- A- Find the electric field \vec{E} in the three regions by first calculating \vec{D} .
- B- Find the bound charge ρ_b in the dielectric and σ_b on the outer surface of the shell.

Answer:

A- Due to the spherical symmetry, we can use Gauss' law in the form:

$$\oint \vec{\nabla} \cdot \vec{D} \, d\tau \equiv \oint \vec{D} \cdot d\vec{a} = \oint \rho_f \, d\tau = Q$$

$$\Rightarrow \vec{D} = \frac{1}{4\pi} \frac{Q}{r^2} \hat{r}$$
$$\vec{E} = \frac{1}{\varepsilon_o} \left(\vec{D} - \vec{P} \right)$$
$$r < a: \quad \vec{P} = 0 \quad \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon_o} = k \frac{Q}{r^2} \hat{r}$$
$$a < r < b: \quad \vec{P} = c \vec{r} \quad \Rightarrow \vec{E} = k \frac{Q}{r^2} \hat{r} - \frac{c}{\varepsilon_o} \vec{r}$$
$$b < r: \quad \vec{P} = 0 \quad \Rightarrow \vec{E} = \frac{\vec{D}}{\varepsilon_o} = k \frac{Q}{r^2} \hat{r}$$

B-

$$\rho_{b} = -\nabla . \vec{P} = -\frac{1}{r^{2}} \frac{\partial}{\partial r} (r^{2} cr) = -3c$$
$$\sigma_{b} = \vec{P} . \hat{n} = \begin{cases} -\vec{P} . \hat{r} \Big|_{r=a} = -ca \\ \vec{P} . \hat{r} \Big|_{r=b} = cb \end{cases}$$

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