SPHERICAL COORDINATES

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \qquad (1)$$

First assume **azimuthal** symmetry $\Rightarrow$ $\phi=\text{constant}$ $\Rightarrow$ $V\equiv V(r,\theta)$, then equation (1) reduces to:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0. \qquad (2)$$

Using the potential in the separated form as:

$$V(r,\theta)=R(r)\Theta(\theta) \qquad (3)$$

equation (2) will be:

$$\underbrace{\overbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)}^{\text{Radial equation}}}_{\ell(\ell+1)}+\underbrace{\overbrace{\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)}^{\text{Angular equation}}}_{-\ell(\ell+1)}=0. \qquad (4)$$

The general solution of the radial equation is:

$$R(r)=\underbrace{Ar^{\ell}}_{\text{regular?}}+\frac{B}{\underbrace{r^{\ell+1}}_{\text{irregular?}}} \qquad (5)$$

H.W. check that equation (5) is a solution of the radial equation.

The general solution of the angular equation is (take it for granted):

$$\Theta(\theta)=\underbrace{P_l(\cos\theta)}_{\text{Legender polynomials}} \qquad (5)$$

Finally the general solution of the potential will be:

$$\begin{aligned}V(r,\theta)&=\left(Ar^{\ell}+\frac{B}{r^{\ell+1}}\right)P_\ell(\cos\theta)\\ &\equiv\sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell}+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)\end{aligned} \qquad (6)$$

H.W. a- Derive $P_3(x)$ and $P_1(x)$ from the formula $P_\ell(x)=\frac{1}{2^\ell\ell!}\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell$,

b- and calculate the integral $\int_{-1}^{1}P_1(x)P_3(x)dx$

Useful integral: $$\int_{-1}^{1}P_\ell(x)P_{\ell'}(x)dx=\begin{cases}0 & \text{if } \ell'\neq\ell\\ \dfrac{2}{2\ell+1} & \text{if } \ell'=\ell\end{cases}$$

Example: The potential at the surface of a sphere (radius R) is given by:

$$V\ (R,\theta) = V_o(\theta) = k\ \sin^2(\theta/2)$$

where *k* is a constant.

a- Express $V_o(\theta)$ in terms of the Legendre Polynomials $P_l(x), x = \cos\theta$.

b- If the potential inside the sphere is given by the equation:

$$V\ (r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)\ .$$

Calculate the allowed values of B_l.

c- If the potential outside the sphere is given by the equation:

$$V\ (r,\theta) = \sum_{\ell=0}^{\infty}\left(\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)\ .$$

Calculate the allowed values of B_l.

Ans:

a-

$$\begin{aligned} V_o(R,\theta) &= k\ \sin^2(\theta/2) = \frac{k}{2}\left[1-\cos(\theta)\right] \\ &= \frac{k}{2}\left[P_o(x) - P_1(x)\right] \end{aligned} \qquad \text{(a)}$$

b- At the surface of the sphere we require that:

$$\begin{aligned} V\ (R,\theta) &= \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)\Big|_{r=R} \\ &= \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = A_0 P_0(\cos\theta) + A_1 R P_1(\cos\theta) + \cdots \end{aligned} \qquad \text{(b)}$$

By comparing equations (a) and (b) one finds:

$$A_0 = \frac{k}{2}, A_1 = -\frac{k}{2R}$$

and all other A_ℓ 's vanish. Evidently,

$$V\ (R,\theta) = \frac{k}{2}\left[P_0(\cos\theta) - \frac{r}{R}P_1(\cos\theta)\right] = \underline{\frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)}$$

c- At the surface of the sphere we require that: $V(R,\theta)=\sum_{\ell=0}^{\infty}\left(\frac{B_\ell}{R^{\ell+1}}\right)P_\ell(\cos\theta)=V_o(\theta)$.

Multiplying by $P_{\ell'}(\cos\theta)\sin\theta$ and integrating, and use the orthogonality relation, we have $B_\ell=\frac{2\ell+1}{2}R^{\ell+1}\int_o^\pi V_o(\theta)P_\ell(\cos\theta)\sin\theta d\theta$. From $V_o(\theta)=\frac{k}{2}\left[P_o(x)-P_1(x)\right]$, only $\ell=0$ and 1 will be survived and the rest are zero. So, $B_0=\frac{k}{4}R\times 2=k\frac{R}{2}$, and $B_1=-\frac{3}{2}R^2\times\frac{2k}{6}=-k\frac{R^2}{2}$. Finally,

$$V(R,\theta)=\underline{\frac{k}{2}\frac{R}{r}\left(1-\frac{R}{r}\cos\theta\right)}$$