# SPHERICAL COORDINATES

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0. \quad (1)$$

First assume **azimuthal** symmetry $\Rightarrow$ $\phi = \text{constant}$ $\Rightarrow$ $V \equiv V(r,\theta)$, then equation (1) reduces to:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0. \quad (2)$$

Using the potential in the separated form as:

$$V(r,\theta)=R(r)\Theta(\theta) \quad (3)$$

equation (2) will be:

$$\underbrace{\overbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)}^{\text{Radial equation}}}_{\ell(\ell+1)}+\underbrace{\overbrace{\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)}^{\text{Angular equation}}}_{-\ell(\ell+1)}=0. \quad (4)$$

The general solution of the radial equation is:

$$R(r)=\underbrace{Ar^{\ell}}_{\text{regular?}}+\frac{B}{\underbrace{r^{\ell+1}}_{\text{irregular?}}} \quad (5)$$

**H.W.** check that equation (5) is a solution of the radial equation.

The general solution of the angular equation is (take it for granted):

$$\Theta(\theta)=\underbrace{P_l(\cos\theta)}_{\text{Legender polynomials}} \quad (5)$$

Finally the general solution of the potential will be:

$$\begin{aligned}V(r,\theta)&=\left(Ar^{\ell}+\frac{B}{r^{\ell+1}}\right)P_{\ell}(\cos\theta)\\&\equiv\sum_{\ell=0}^{\infty}\left(A_{\ell}r^{\ell}+\frac{B_{\ell}}{r^{\ell+1}}\right)P_{\ell}(\cos\theta)\end{aligned} \quad (6)$$

**H.W.** a- Derive $P_3(x)$ and $P_1(x)$ from the formula $P_{\ell}(x)=\frac{1}{2^{\ell}\ell!}\frac{d^{\ell}}{dx^{\ell}}\left(x^2-1\right)^{\ell}$,

b- and calculate the integral $\int_{-1}^{1}P_1(x)P_3(x)dx$

Useful integral:

$$\int_{-1}^{1}P_{\ell}(x)P_{\ell'}(x)dx=\begin{cases}0 & \text{if } \ell'\neq\ell\\ \dfrac{2}{2\ell+1} & \text{if } \ell'=\ell\end{cases}$$

# Examples

7- Two infinite concentric cylindrical metal shell, with radii "$a$" and "$b$", $b > a$. The inner cylinder has a linear charge density "$\lambda$" and the outer cylinder has "$-\lambda$". Find

a- the electric field at $s$, $a < s < b$,

b- the potential difference between the two radii,

c- the capacitance of the system,

d- the energy stored in the system.

**Ans:**

a- Use Gauss' theorem

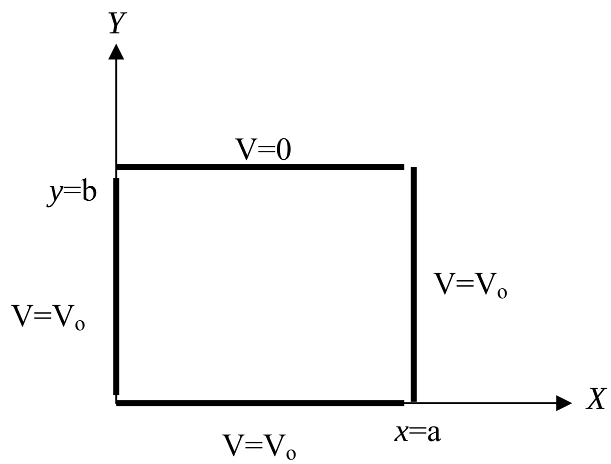
$$\phi = \int E.da = \frac{Q_{in}}{\varepsilon_o} \Rightarrow E(2\pi s l) = \frac{\lambda l}{\varepsilon_o} \Rightarrow E = \frac{2k\lambda}{s},\ k = \frac{1}{4\pi\varepsilon_o}$$

b-
$$V = \int_a^b E ds = 2k\lambda \left[\ln s\right]^b_a = 2k\lambda \ln\left(\frac{a}{b}\right)$$

c-
$$C = Q/V = \lambda l / 2k\lambda \ln\left(\frac{b}{a}\right) = \frac{l}{2k\ln\left(\frac{b}{a}\right)}$$

d-
$$W = \frac{\varepsilon_o}{2}\int |E|^2 d^3s = \frac{1}{8\pi}\int \left|\frac{2k\lambda}{s}\right|^2 s ds d\varphi dz =$$
$$= \lambda^2 k^2 l \ln\left(\frac{b}{a}\right) = \frac{1}{2}QV$$

1- A duct is infinitely long in the $z$ direction and has a rectangular cross section bounded by the planes $x = 0$, $x = a$, $y = 0$, and $y = b$. The plane $y = b$ is at zero potential, and the other planes are at potential $V_o$. With two different methods, find the potential everywhere inside the duct.




**Answer: B.C. required:**

$V(0,y) = V_o$ (i)

$V(a,y) = V_o$ (ii)

$V(x,0) = V_o$ (iii)

$V(x,b) = 0$ (iv)

Try the trigonometric function in x-direction and the exponential in y-direction.

$$V(x,y) = Ax + By + C + \sum_k^\infty a_k \left[\cos(kx) + b_k \sin(kx)\right]\left[c_k \cosh(ky) + d_k \sinh(ky)\right]$$

**B.C. (i) required that:**

$$V(0,y) \equiv V_o = 0 + By + C + \sum_k^\infty a_k \left[c_k \cosh(ky) + d_k \sinh(ky)\right]$$

which implies:

$$B = 0; \quad C = V_0; \quad \text{and} \quad a_k = 0$$

$$\therefore \quad V(x,y) = Ax + V_o + \sum_k^{\infty} \sin(kx)\left[c'_k \cosh(ky) + d'_k \sinh(ky)\right]$$

**B.C. (ii) required that:**

$$V(a,y) \equiv V_o = Aa + V_o + \sum_k^{\infty} \sin(ka)\left[c'_k \cosh(ky) + d'_k \sinh(ky)\right]$$

which implies:

$$A = 0; \quad C = V_0; \quad \text{and} \quad ka = m\pi, \qquad n = 1,2,3,\cdots$$

Therefore $V(x,y)$ reduces to:

$$V(x,y) = V_o + \sum_{n=1,2,}^{\infty} \sin(\frac{n\pi}{a}x)\left[c'_k \cosh(\frac{n\pi}{a}y) + d'_k \sinh(\frac{n\pi}{a}y)\right]$$

**B.C. (iii) required that:**

$$V(x,0) \equiv V_o = V_o + \sum_{n=1,2,}^{\infty} \sin(\frac{n\pi}{a}x)\left[c'_k\right] \quad \Rightarrow \quad c'_k = 0$$

then

$$V(x,y) = V_o + \sum_{n=1,2,}^{\infty} d'_k \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$$

**B.C. (iv) required that:**

$$-V_o = \sum_{n=1,2,}^{\infty} d'_k \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}b) \quad \Rightarrow \quad c'_k = 0$$

To calculate the constant $d'_k$, we have to use the Fourier's trick. Multiply the above equation with $\sin(\frac{m\pi}{a}x)$ and integrate over $x$ :

$$\underbrace{-V_o \int_0^a \sin(\frac{m\pi}{a}x)dx}_{\frac{V_o a}{m\pi}\left[1-(-1)^m\right]} = \sum_{n=1,2,}^{\infty} d'_k \sinh(\frac{n\pi}{a}b) \underbrace{\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx}_{=\frac{a}{2}\delta_{mn}}$$

$$\therefore V(x,y) = V_o - \frac{4V_o}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin(\frac{n\pi}{a}x) \frac{\sinh(\frac{n\pi}{a}y)}{\sinh(\frac{n\pi}{a}b)}$$

# Dipole

**Problem 3.27** Four particles (one of charge $q$, one of charge $3q$, and two of charge $-2q$) are placed as shown in Fig. 3.31, each a distance $a$ from the origin. Find a simple approximate formula for the potential, valid at points far from the origin. (Express your answer in spherical coordinates.)

$$V(r) = V_{monopole} + V_{dipole};$$

$$V_{monopole} = k\,\frac{Q}{r};$$

$$V_{dipole} = k\,\frac{\vec{p}.\hat{r}}{r^2}.$$

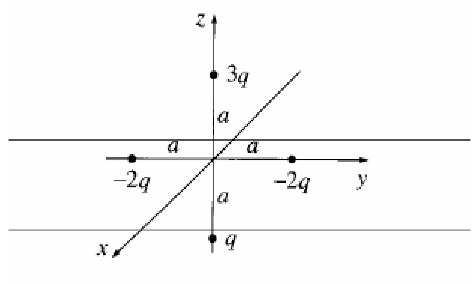



Figure 3.31

Where

$$Q = \sum_{i=1}^{5} q_i = 0;$$

$$\vec{p} = \sum_{i=1}^{5} q_i r_i = (-2q)(-a)\,\hat{\jmath} - 2qa\,\hat{\jmath} - q(-a)\,\hat{k} + 3qa\,\hat{k} = 2qa\,\hat{k};$$

$$\vec{p}.\hat{r} = 2qa(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\cdot\hat{r} = 2qa\cos\theta.$$

Then

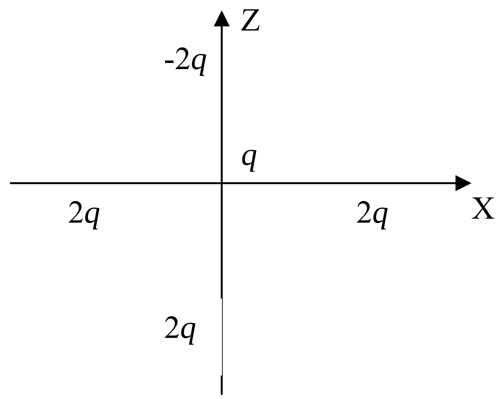
$$V_{dipole} = k\,\frac{\vec{p}.\hat{r}}{r^2} = \frac{2qa\cos\theta}{r^2}.$$

And

$$V_{total} = k\,\frac{2qa\cos\theta}{r^2}$$

$$\begin{aligned}\vec{E} = -\nabla V &= -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi} \\ &= k\left[\left(\frac{4qa\cos\theta}{r^3}\right)\hat{r} - \frac{1}{r}\frac{2qa\cos\theta}{r^2}\hat{\theta}\right] \\ &= k\left[\left(\frac{4qa\cos\theta}{r^3}\right)\hat{r} + \frac{2qa\cos\theta}{r^3}\hat{\theta}\right]\end{aligned}$$

1- Calculate the potential and the electric field, at a distance $r$ from the configuration shown in the figure. The charge $q$ at the origin and the rest are a distance $d$ from the origin.




**Ans:**

$$V(r) = V_{monopole} + V_{dipole};$$

$$V_{monopole} = k\,\frac{Q}{r};$$

$$V_{dipole} = k\,\frac{\vec{p}.\hat{r}}{r^2}.$$

Where

$$Q = \sum_{i=1}^{5} q_i = 5q;$$

$$\vec{p} = \sum_{i=1}^{5} q_i r_i = 2qd\ \hat{\mathrm{j}} - 2qd\ \hat{\mathrm{j}} - 2qd\ \hat{\mathrm{k}} - 2qd\ \hat{\mathrm{k}} = -4qd\ \hat{\mathrm{k}};$$

$$\vec{p}.\hat{\mathrm{r}} = -4qd\,(\cos\theta\,\hat{\mathrm{r}} - \sin\theta\,\hat{\theta})\hat{\mathrm{r}} = -4qd\,\cos\theta.$$

Then

$$V_{dipole} = k\,\frac{\vec{p}.\hat{r}}{r^2} = -k\,\frac{4qd\,\cos\theta}{r^2}.$$

And

$$V_{total} = k\left(\frac{5q}{r} - \frac{4qd\,\cos\theta}{r^2}\right)$$

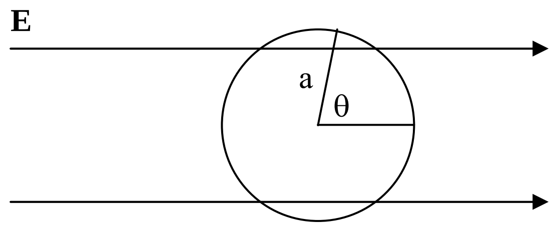
$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathrm{r}} - \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi}\hat{\varphi}$$

$$= k\left[\left(-\frac{5q}{r^2} - \frac{8qd\,\cos\theta}{r^3}\right)\hat{\mathrm{r}} - \frac{1}{r}\frac{4qd\,\cos\theta}{r^2}\hat{\theta}\right]$$

$$= k\left[\left(\frac{5q}{r^2} + \frac{8qd\,\cos\theta}{r^3}\right)\hat{\mathrm{r}} + \frac{4qd\,\cos\theta}{r^3}\hat{\theta}\right]$$

## Spherical symmetry

2- A conducting sphere of radius "a" bearing total charge "Q" is placed in an initially uniform electric field $\vec{E} = E_o\,\hat{z}$. Find the potential at all points exterior to the sphere.




Ans: Start with the expression for the potential in the form:

$$V\,(r,\theta) = \frac{Q}{r} + \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Now, using the B. C, as $r \to \infty$ we wish the potential to behave as $-E_o z = -Er\cos\theta$ Thus $A_1 = -E_o$ and all others $A_n's$ are zero.

$$V(r,\theta) = \frac{Q}{r} - E_o r\cos\theta + \sum_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos\theta)$$

Now, we have to make sure that this potential satisfy the B.C. at $r = a$,

$$V\,(a,\theta) = \frac{Q}{a} = \frac{Q}{a} - E_o a\cos\theta + \sum_{n=0}^{\infty} B_n a^{-n-1} P_n(\cos\theta)$$

$$= -E_o a\cos\theta + \frac{B_o}{a} + \frac{B_1}{a^2}P_1(\cos\theta) + \frac{B_2}{a^3}P_2(\cos\theta) + \cdots$$

$$\Rightarrow B_o = 0, \quad B_1 = E_o a^3, \quad B_n = 0 \quad \text{for } n \neq 0,1.$$

Thus the final solution that satisfy all B. C. is

$$\boxed{V\,(r,\theta) = \left(\frac{a^3}{r^3} - 1\right)E_o r\cos\theta + \frac{Q}{r}.}$$

**1-** The potential at the surface of a sphere (radius R) is given by:

$$V\,(R,\theta) = V_o(\theta) = K\,\sin^2(\theta/2)$$

where $k$ is a constant.

a- Express $V_o(\theta)$ in terms of the Legendre Polynomials $P_l(x)$, $x = \cos\theta$.

b- If the potential inside the sphere is given by the equation:

$$V(r,\theta)=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta).$$

Calculate the allowed values of $A_\ell$.

c- If the potential outside the sphere is given by the equation:

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta).$$

Calculate the allowed values of $B_\ell$.

**Ans:**

a-

$$\begin{aligned}V_o(R,\theta)&=K\sin^2(\theta/2)=\frac{K}{2}\left[1-\cos(\theta)\right]\\&=\frac{K}{2}\left[P_o(x)-P_1(x)\right]\end{aligned}\tag{a}$$

b- At the surface of the sphere we require that:

$$\begin{aligned}V(r,\theta)&=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta)\Big|_{r=R}\\&=\sum_{\ell=0}^{\infty}A_\ell R^\ell P_\ell(\cos\theta)=A_0P_0(\cos\theta)+A_1RP_1(\cos\theta)+\cdots\end{aligned}\tag{b}$$

By comparing equations (a) and (b) one finds:

$$A_0=\frac{K}{2},\quad A_1=-\frac{K}{2R}$$

and all other $A_\ell$'s vanish. Evidently,

$$\begin{aligned}V(R,\theta)&=\frac{K}{2}\left[P_0(\cos\theta)-\frac{r}{R}P_1(\cos\theta)\right]\\&=\underline{\frac{K}{2}\left(1-\frac{r}{R}\cos\theta\right)}\end{aligned}$$

c- At the surface of the sphere we require that: $V(r,\theta)=\sum_{\ell=0}^{\infty}\left(\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)=V_o(\theta)$.

Multiplying by $P_{\ell'}(\cos\theta)\sin\theta$ and integrating, and use the orthogonality relation, we have $B_\ell=\frac{2\ell+1}{2}R^{\ell+1}\int_o^\pi V_o(\theta)P_\ell(\cos\theta)\sin\theta d\theta$.

From

$$V_o(\theta) = \frac{K}{2}\left[P_o(x) - P_1(x)\right],$$

only $\ell = 0$ and 1 will be survived and the rest are zero. So, $B_0 = \frac{K}{4}R \times 2 = K\,\frac{R}{2}$, and $B_1 = -\frac{3}{2}R^2 \times \frac{2K}{6} = -K\,\frac{R^2}{2}$. Finally,

$$V\,(R,\theta) = \underline{\frac{K}{2}\frac{R}{r}\left(1 - \frac{R}{r}\cos\theta\right)}$$

**Example:** Determine the potential of a charged sphere of radius R. The surface charge density varies according to the law $\sigma = \sigma_o \cos\theta$..

**Solution:**

Start with the general solution:

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$V_{out}(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

On the sphere, we can apply the boundary conditions:

$$V_{in}\Big|_{r=R} = V_{out}\Big|_{r=R}$$

$$\left(\frac{\partial V_{in}}{\partial r} - \frac{\partial V_{out}}{\partial r}\right)_{r=R} = \frac{\sigma}{\varepsilon_o}$$

one finds**:**

$$A_1 = \frac{\sigma_o}{3\varepsilon_o}, \quad B_1 = \frac{\sigma_o}{3\varepsilon_o}R^3;$$

$$A_\ell = B_\ell = 0 \quad \text{for} \quad \ell \neq 1$$

Thus

$$V_{in} = \frac{\sigma_o}{3\varepsilon_o} r\cos\theta,$$

$$V_{out} = \frac{\sigma_o}{3\varepsilon_o}\frac{R^3}{r^2}\cos\theta$$

**Example:** The potential at the surface of a sphere of radius R is given by $V\left(\theta\right)=V_o\cos\left(2\theta\right)$ , where $V_o$ is a constant..

a- Show that $V=\frac{V_o}{3}(4P_2-P_o)$

b- Using the previous result find the potential inside and outside the sphere assuming there is no charge inside or outside the sphere.

**Solution:**

a- It is easy to prove it by expanding $V$ in terms of Legendre polynomial in the form:

$$V=\alpha P_2(\cos\theta)+\beta P_o(\cos\theta)$$

One finds, $\alpha=\frac{4V_o}{3}, \quad \beta=-\frac{V_o}{3}$

b- Start with the general solution:

$$V_{in}(r,\theta)=\sum_{\ell=0}^{\infty}A_\ell r^\ell P_\ell(\cos\theta) \qquad r<R$$

$$V_{out}(r,\theta)=\sum_{\ell=0}^{\infty}\frac{B_\ell}{r^{\ell+1}}P_\ell(\cos\theta) \qquad r>R$$

On the sphere, we can apply the boundary conditions:

$$V(R,\theta)=V_O\left(\theta\right)=\frac{V_o}{3}(4P_2-P_o)$$

$$\sum_{\ell=0}^{\infty}A_\ell R^\ell P_\ell(\cos\theta)=\frac{V_o}{3}(4P_2-P_o)$$

Using Fourier's trick: multiply by $P_m(\cos\theta)\sin\theta\, d\theta$ and integrate from 0 to $2\pi$ , or simply expand the left hand side of the above equation and equate the same power of $P_\ell$, e.g.

$$A_o+A_1RP_1(\cos\theta)+A_2R^2P_2(\cos\theta)+\cdots=\frac{V_o}{3}(4P_2-P_o)$$

$$A_2=\frac{4V_o}{3R^2}, \quad A_0=-\frac{V_o}{3};$$
$$A_\ell=0 \quad \text{for} \quad \ell\neq 0,2$$

Thus:

$$V_{in}(r,\theta)=\frac{4}{3}\frac{V_o}{R^2}r^2(\frac{3}{2}\cos^2\theta-\frac{1}{2})-\frac{1}{3}V_o \qquad r<R$$

For $r>R$, the continuity at the boundary condition $R$ required that: $B_\ell = A_\ell R^{2\ell+1}$, thus:

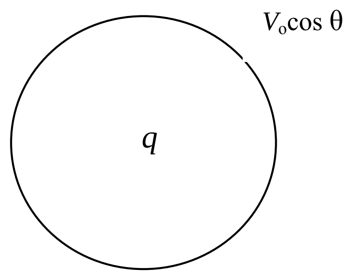
$$B_2=\frac{4V_o}{3}R^3, \quad B_0=-\frac{V_o}{3}R;$$
$$B_\ell=0 \quad \text{for} \quad \ell\neq 0,2$$

$$V_{out}(r,\theta)=\frac{4}{3}V_oR^3\frac{1}{r^3}(\frac{3}{2}\cos^2\theta-\frac{1}{2})-\frac{1}{3}V_o\frac{R}{r} \qquad r>R$$

5- A spherical surface of radius "$a$" has its center of symmetry at the origin of spherical coordinates and has a potential distribution $V_o\cos\theta \equiv V_oP_1(\cos\theta)$ maintained on it.

a- If a point charge "$q$" is located at the origin, find the potential for $r\leq a$.

b- If we remove the point charge, find the potential inside and outside the sphere.




**Answer:** Start with the general potential:

$$V(r,\theta)=\sum_{n=0}^{\infty}\left(A_nr^n+\frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Now, as $r\to 0$ we wish the potential to behave as $k\frac{q}{r}$. Thus $B_o=kq$ and all others $B_n$'s are zero.

$$V(r,\theta)=k\frac{q}{r}+\sum_{n=0}^{\infty}A_nr^nP_n(\cos\theta)$$

Now, we have to make sure that this potential satisfy the B.C. at $r=a$,

$$V(a,\theta)=V_o\cos\theta=V_oP_1(\cos\theta)=k\frac{q}{a}+A_o+A_1aP_1(\cos\theta)+A_2a^2P_2(\cos\theta)+\cdots$$

$$\Rightarrow A_1a=V_o,$$

and $\quad k\frac{q}{a}+A_o=0 \quad\Rightarrow\quad A_o=-k\frac{q}{a}, \qquad A_n=0$ for $n\geq 2$

Thus the final solution that satisfy all B. C. is

$$V(r,\theta) = kq\left(\frac{1}{r} - \frac{1}{a}\right) + \frac{V_o}{a} r\cos\theta.$$

b- In case of q not exist, one finds:

Inside the sphere

$$A_1 = A/a, \qquad A_\ell = 0 \quad \text{for } \ell \neq 1$$

$$\Rightarrow \quad V_{in}(r,\theta) = \frac{V_o}{a} r\cos\theta,$$

Outside the sphere

$$B_1 = Aa^2, \qquad B_\ell = 0 \quad \text{for } \ell \neq 1$$

$$\Rightarrow \quad V_{ou}(r,\theta) = \frac{V_o a^2}{r^2}\cos\theta$$

## Dielectric

$$\sigma_b = \vec{P}.\hat{n}, \quad \rho_b = -\vec{\nabla}.\vec{P},$$

$$\rho_{total} = \underbrace{\rho_b}_{\text{bound chargees}} + \underbrace{\rho_f}_{\text{free chargees}}$$

$$\vec{\nabla}.\vec{E} = \frac{\rho_{total}}{\varepsilon_o}$$

$$\Rightarrow \quad \varepsilon_o \vec{\nabla}.\vec{E} = \rho_b + \rho_f = -\vec{\nabla}.\vec{P} + \rho_f ,$$

$$\Rightarrow \quad \vec{\nabla}. \underbrace{\varepsilon_o\left(\vec{E}+\vec{P}\right)}_{\vec{D}\,=\,\text{Displacement current}} = \rho_f$$

### Gauss' law:

$$\oint \vec{\nabla}.\vec{D}\, d\tau \equiv \oint \vec{D}.d\vec{a} = \oint \rho_f \, d\tau = Q_f$$

$$\vec{P} = \varepsilon_o \chi_e \vec{E}$$

$\chi_e \equiv$ Electric susceptibility of the medium (measure the response to the external filed (magnetic, electric, etc. )

$\varepsilon_o \equiv$ Permittivity of free space.

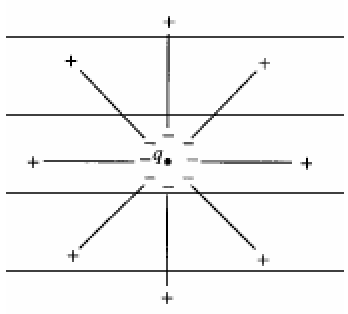
$$\vec{D} = \varepsilon_o\left(\vec{E}+\vec{P}\right) = \varepsilon_o\vec{E} + \varepsilon_o\chi_e\vec{E} = \underbrace{\varepsilon_o\left(1+\chi_e\right)}_{\varepsilon}\vec{E}$$

$$= \varepsilon\,\vec{E}$$

$\varepsilon \equiv$ Permittivity of the material.

$\kappa = \varepsilon_r = \dfrac{\varepsilon}{\varepsilon_o} = 1 + \chi_e \equiv$ Relative permittivity or dielectric constant.

**Example:** Consider a free point charge $Q_f$ is embedded at the center of a dielectric. Find $\vec{D}$, $\vec{E}$ $\vec{P}$ and the polarization charge densities $\rho_b$ and $\sigma_b$.




$$\oint \vec{\nabla} \,.\, \vec{D}\; d\tau \equiv \oint\; \vec{D}\,.\, d\vec{a} = \oint \rho_f \;\; d\tau = Q_f \qquad (1)$$

But

$$\oint\; \vec{\mathrm{D}}.\, \mathrm{d}\vec{a} = D\,(4\pi r^2) \qquad (2)$$

From (1) and (2)

$$D = \frac{1}{4\pi}\frac{Q_f}{r^2} \quad \Rightarrow \text{ from the symmetry of the problem } \vec{D} = \frac{1}{4\pi}\frac{Q_f}{r^2}\hat{\mathrm{r}},$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{1}{4\pi\varepsilon}\frac{Q_f}{r^2}\hat{\mathrm{r}}$$

Since:

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = \varepsilon_o \vec{E} + \varepsilon_o \chi_e \vec{E} = \underbrace{\varepsilon_o\left(1+\chi_e\right)}_{\varepsilon}\vec{E} \;\; = \;\; \underline{\varepsilon\,\vec{E}}$$

Now,

$$\vec{P} = \frac{\vec{D} - \varepsilon_o \vec{E}}{\varepsilon_o} = \left(\varepsilon - 1\right)\vec{E} = \frac{\left(\varepsilon-1\right)}{4\pi\varepsilon}\frac{Q_f}{r^2}\hat{\mathrm{r}},$$

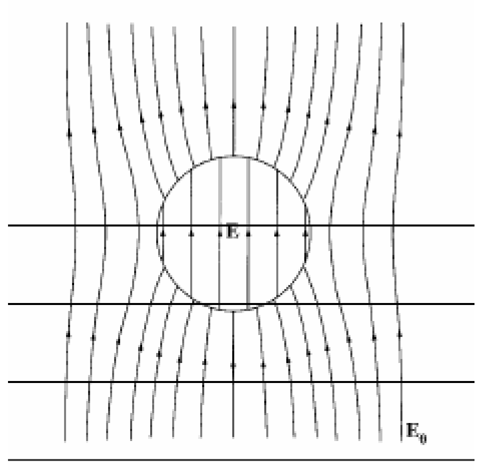
And

$$\rho_b = -\nabla.\vec{P} = - = \frac{\left(\varepsilon-1\right)}{4\pi\varepsilon}Q_f \;\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{1}{r^2}\right) = 0,$$

$$Q_b = \sigma_b \;\text{ Area} = -\left(\frac{\varepsilon-1}{\varepsilon}\right)Q_f$$

$$\text{effective charge } = \; Q_b + Q_f \;\;\; = \frac{Q_f}{\varepsilon}$$

**Example:** A sphere of homogeneous linear dielectric material is placed in a uniform external electric field $\vec{E}_o$ . Find the potential inside and outside the sphere.




**Solution:**

Start with the general solution:

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$\overbrace{\qquad\text{Inside}\qquad\qquad\qquad\text{Outside}\qquad}^{V_\ell(r,\theta)=\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)}$$

$$\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)_I P_\ell(\cos\theta)\qquad\qquad\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)_{II} P_\ell(\cos\theta)$$

**B. C.** $V(\infty,\theta)=-E_o z=-E_o r\cos\theta \Rightarrow \ell=1,\quad A_{II}=-E_o$

**B. C.** $V(0,\theta)$ must be finite $\Rightarrow\quad B_I=0$

$$\overbrace{\qquad\text{Inside}\qquad\qquad\qquad\text{Outside}\qquad}^{V(r,\theta)}$$

$$A_I r\cos\theta\qquad\qquad\left(-E_o r+\frac{B_{II}}{r^2}\right)\cos\theta$$

$$E_{I\theta}=E_{II\theta}$$

**B. C.**
$$\left(-\frac{1}{r}\frac{\partial V_I}{\partial\theta}\right)_{r=a}=\left(-\frac{1}{r}\frac{\partial V_{II}}{\partial\theta}\right)_{r=a}$$

$$\Rightarrow \quad A_I = -E_o + \frac{1}{a^3} B_{II} \tag{1}$$

**B. C.**

$$D_{In} = D_{IIn}$$
$$\varepsilon_1 E_{Ir} = \varepsilon_2 E_{IIr}$$
$$\left(-\varepsilon_1 \frac{\partial V_I}{\partial r}\right)_{r=a} = \left(-\varepsilon_2 \frac{\partial V_{II}}{\partial \theta}\right)_{r=a}$$

$$\Rightarrow \quad \varepsilon_1 A_I = -\varepsilon_2 E_o - \frac{2\varepsilon_2}{a^3} B_{II} \tag{2}$$

Solving equations (1) and (2) gives:

$$A_I = -\left(\frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right) E_o, \quad B_{II} = \left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right) a^3 E_o$$

And

$$\overbrace{\text{Inside} \qquad\qquad\qquad \text{Outside}}^{V(r,\theta)}$$

$$-\left(\frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right) E_o r \cos\theta \qquad\qquad \left\{\left(\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2}\right)\frac{a^3}{r^2} - r\right\} E_o \cos\theta$$

$$V_{in} = -\left(\frac{3\varepsilon_2}{\varepsilon_1 + 2\varepsilon_2}\right) E_o r \cos\theta = -\left(\frac{3\varepsilon_2 E_o}{\varepsilon_1 + 2\varepsilon_2}\right) r \cos\theta$$
$$= -\left(\frac{3\varepsilon_2 E_o}{\varepsilon_1 + 2\varepsilon_2}\right) Z$$

**Hence the field inside the sphere is:**

$$E_{in} = \left(\frac{3\varepsilon_2 E_o}{\varepsilon_1 + 2\varepsilon_2}\right),$$

**and is uniform**

**Example:** A spherical dielectric of radius "*R*" and dielectric constant $\kappa = \dfrac{\varepsilon}{\varepsilon_o}$ has a uniform free charge density $\rho_o$ distributed through it. Find

a- The electric displacement inside and outside the sphere.

b- The electric field inside and outside the sphere.

c- The electric potential at the center of the sphere.

d- The induced charge density.

**Ans:**

a- Inside the sphere $(r < a)$: draw a Gaussian surface with radius *r* and then calculate the total free charges inside the Gaussian surface using Gauss' law:

$$\underbrace{\oint \vec{D}_{in} . d\vec{a}}_{D_{in}(4\pi r^2)} = \underbrace{Q_f}_{\frac{4}{3}\pi r^3 \rho_o} \equiv \text{The total free charges inside the Gaussian surface}$$

$$\therefore \vec{D}_{in}(\vec{r}) = \frac{1}{3}\rho_o r \ \hat{\text{r}};$$

$$\vec{E}_{in} = \frac{\vec{D}_{in}}{\varepsilon} = \frac{1}{3}\frac{\rho_o}{\varepsilon_o \kappa} r \ \hat{\text{r}}, \qquad \kappa = \frac{\varepsilon}{\varepsilon_o}$$

b- Outside the sphere $(r > a)$: draw a Gaussian surface with radius *r* and then calculate the total free charges inside the Gaussian surface using Gauss' law:

$$\underbrace{\oint \vec{D}_{out} . d\vec{a}}_{D_{out}(4\pi r^2)} = \underbrace{Q_f}_{\frac{4}{3}\pi R^3 \rho_o} \equiv \text{The total free charges inside the Gaussian surface}$$

$$\therefore \vec{D}_{out}(\vec{r}) = \frac{1}{3r^2}\rho_o R^3 \ \hat{\text{r}};$$

$$\vec{E}_{out} = \frac{\vec{D}_{out}}{\varepsilon_o} = \frac{1}{3}\frac{\rho_o}{\varepsilon_o r^2} R^3 \ \hat{\text{r}},$$

c-

$$V(0) = -\int_{\infty}^{0} \vec{E}.d\vec{r} = -\int_{\infty}^{R} \vec{E}_{out}.d\vec{r} - \int_{R}^{0} \vec{E}_{in}.d\vec{r}$$
$$= -\frac{1}{3}\frac{\rho_o}{\varepsilon_o}R^3\int_{\infty}^{R}\frac{1}{r^2}dr - \frac{1}{3}\frac{\rho_o}{\varepsilon_o \kappa}R^3\int_{R}^{0} rdr$$
$$= \frac{1}{3}\frac{\rho_o}{\varepsilon_o}R^3\left(1+\frac{1}{2\kappa}\right)$$

To know the bound charge densities, we need to calculate $\vec{P}(\vec{r})$. To calculate $\vec{P}(\vec{r})$, we use $\vec{D}_{in}$ and $\vec{E}_{in}$.

$$\vec{D}_{in} = \varepsilon_o \vec{E}_{in} + \vec{P}_{in} = \varepsilon_o \vec{E}_{in} + \varepsilon_o \chi_e \vec{E}_{in} = \underbrace{\varepsilon_o\left(1+\chi_e\right)}_{\varepsilon}\vec{E}_{in} = \underline{\varepsilon \, \vec{E}_{in}}$$

Now,

$$\vec{P}_{in} = \varepsilon_o \chi \vec{E}_{in} = \varepsilon_o(\kappa-1)\vec{E}_{in} = \frac{1}{3}\rho_o\frac{\kappa-1}{\kappa} r\,\hat{\text{r}} = \frac{1}{3}\rho_o\frac{\kappa-1}{\kappa}\vec{r},$$

And

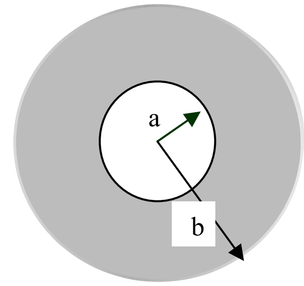
$$\rho_b = -\nabla.\vec{P} = -\frac{1}{3}\rho_o\frac{\kappa-1}{\kappa}\underbrace{\nabla.\vec{r}}_{\nabla\cdot\vec{r}=\frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3} = \underline{-\rho_o\frac{\kappa-1}{\kappa}}$$

$$\therefore \rho = -\rho_o\frac{\kappa-1}{\kappa} \text{ (throught the sphere).}$$

$$\sigma_b = \vec{P}.\hat{\text{n}} = \vec{P}.\hat{\text{r}}\Big|_{r=a} = \underline{\frac{1}{3}\rho_o\frac{\kappa-1}{\kappa}a}\,.$$

Note that: $\kappa = \varepsilon_r = \dfrac{\varepsilon}{\varepsilon_o} = 1 + \chi_{\text{e}}$

**Example:** A Conducting wire, carrying a charge $\lambda$ per unit length, is embedded along the axis of a circular cylinder of linear dielectric. The radius of the wire is "*a*" and the radius of the dielectric is "*b*", $b > a$.




a- Show that the bund charge (total charge) on the outer surface of the dielectric is equal to the bound charge on the inner surface except for sign.

b- Show that the net charge along the axis is $\frac{\lambda}{\kappa}$ per unit length.

c- Show that the volume density of bound charge is zero in the dielectric.

d- What is the surface charge density if the outer surface connected to the ground?

**Ans:**

a- In general Gauss' law implies:

$$\underbrace{\oint \vec{D}.d\vec{a}}_{D(2\pi s\ell)} = \underbrace{Q_f}_{\lambda\ell} \quad \Rightarrow \quad \vec{D} = \frac{\lambda}{2\pi s}\,\hat{s}$$

$$\therefore\ \vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\lambda}{2\pi\varepsilon_o \kappa s}\ \hat{s},$$

$$\vec{P} = \varepsilon_o(\kappa-1)\vec{E} = \frac{\lambda(\kappa-1)}{2\pi\kappa s}\ \hat{s}$$

Surface charge density on the outer surface:

$$\sigma_{bo} = \vec{P}.\hat{n} = \vec{P}.\hat{r}\Big|_{s=b} = \frac{\lambda(\kappa-1)}{2\pi\ \kappa}\frac{1}{b},$$

$$Q_{bo} = \int \sigma_{bo}da = \sigma_{bo}(2\pi b\,\ell) = \frac{\lambda\ell(\kappa-1)}{\kappa}$$

Surface charge density on the inner surface:

$$\sigma_{bi} = \vec{P}.\hat{n} = -\vec{P}.\hat{r}\Big|_{s=a} = -\frac{\lambda(\kappa-1)}{2\pi\ \kappa}\frac{1}{a},$$

$$Q_{bi} = \int \sigma_{bi}da = \sigma_{bi}(2\pi a\ell) = -\frac{\lambda\ell(\kappa-1)}{\kappa}$$

$$\therefore Q_{bi} = -Q_{bo} \qquad \text{Q.E.D.}$$

b- The net charge along the axis::

$$Q_t = \lambda\ell - \frac{\lambda\ell(\kappa-1)}{\kappa} = \frac{\lambda\ell}{\kappa} \quad \Rightarrow \quad \lambda' = \frac{Q_t}{\ell} = \frac{\lambda}{\kappa} \qquad \text{Q.E.D.}$$

c-

$$\rho_b = -\nabla.\vec{P} = -\frac{\lambda\ell(\kappa-1)}{2\pi\,\kappa}\frac{1}{s^2} + \frac{\lambda\ell(\kappa-1)}{2\pi\,\kappa}\frac{1}{s^2} = 0$$

d- if the outer surface connected to the ground, the surface charge $\sigma_{bo}$ will be zero, consequently $Q_{bo} = 0$.

**Problem 4.15** A thick spherical shell (inner radius $a$, outer radius $b$) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\,\hat{\mathbf{r}},$$

where $k$ is a constant and $r$ is the distance from the center (Fig. 4.18). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:

(a) Locate all the bound charge, and use Gauss's law (Eq. 2.13) to calculate the field it produces.

(b) Use Eq. 4.23 to find **D**, and then get **E** from Eq. 4.21. [Notice that the second method is much faster, and avoids any explicit reference to the bound charges.]

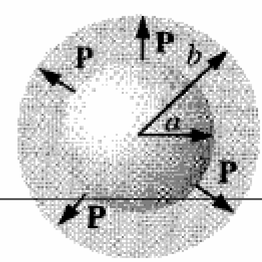



Figure 4.18

**Ans:**

$$\vec{P}(\vec{r}) = \frac{K}{r}\,\hat{\text{r}}$$

$$\rho_b = -\nabla.\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{K}{r}) = -\frac{K}{r^2}$$

$$\sigma_b = \vec{P}.\hat{n} = \begin{cases} \vec{P}.\hat{r}\Big|_{r=b} = \dfrac{K}{b} \\ -\vec{P}.\hat{r}\Big|_{r=a} = -\dfrac{K}{a} \end{cases}$$

For $r < a$ and $r > b$ the electric field will be Zero, since the enclosed charge are zero.

In the region $a < r < b$, the total charge will be:

$$\begin{aligned} Q_{total} &= -\frac{K}{a}(4\pi a^2) + \int_a^r \left(-\frac{K}{r'^2}\right) 4\pi r'^2 dr' \\ &= -4\pi K r \end{aligned}$$

From Gauss' law

$$\underbrace{\oint \vec{E}.d\vec{a}}_{E(4\pi r^2)} = \frac{Q_{total}}{\varepsilon_o} \quad \Rightarrow \quad \vec{E} = -\frac{K}{\varepsilon_o r}\hat{r}$$

b- $\because \oint \vec{D}.d\vec{a} = Q_f = 0 \quad \Rightarrow \quad D = 0$ everywhere

In the region $a < r < b$ :

$$\because \vec{D} = \varepsilon_o \vec{E} + \vec{P} = 0 \quad \Rightarrow \quad \vec{E} = -\frac{1}{\varepsilon_o}\vec{P} = -\frac{K}{\varepsilon_o r}\hat{r}$$