KING FAHD UNIVERSITY OF PETROLUIM AND MINERALS
DEPARTMENT OF PHYSICS

PHYS 305
ELECTRICITY AND MAGNETISM I
SPRING 20082
FIRST MAJOR (2/4/20082)
TIME (6:30 -> 8:30 PM)

Answer the following problems. (SHOW YOUR WORK)

1- In Cartesian coordinates, let $T = xy^2$ and take point "a" to be the origin (0,0,0) and "b" the point (3,3,0). Calculate the value of the integral $\int_a^b \vec{\nabla}T.\, d\vec{l}$ by three different methods.

Ans:

$$I - \quad \int_{a\,P}^{b} \vec{\nabla}\cdot d\,\vec{l} = T\,(b) - T\,(a) = xy^2\Big|_{(0,0,0)}^{(3,3,,0)} = 3\times 3^2 - 0 = \underline{27}.$$

$$II - \quad \vec{\nabla}T\,.d\,\vec{l} = (y^2\;\hat{\text{i}} + 2xy\;\hat{\text{j}}).(dx\;\hat{\text{i}} + dy\;\hat{\text{j}}) = (y^2 + 2xy)dy = 3y^3 dy,$$

we used $y = x$, $dy = dx$ in the last step

$$\therefore \int_a^b \vec{\nabla}T\,.d\,\vec{l} = \int_0^3 3y^2 dy = y^3\Big|_0^3 = \underline{27}$$

$$III - a(0,0,0) \to c(3,0,0) \Rightarrow dx = dy = 0 \Rightarrow \int_a^b \vec{\nabla}T\,.d\,\vec{l} = 0$$

$$c(3,0,0) \to b(3,3,0) \Rightarrow x = 3, dx = 0 \Rightarrow \vec{\nabla}T\,.d\,\vec{l} = 6ydy$$

$$\Rightarrow \int_b^c \vec{\nabla}T\,.d\,\vec{l} = \int_0^3 6ydy = 3y^2\Big|_0^3 = \underline{27}.$$

2- The region $r \le a$, where a is the radius of the sphere, in spherical coordinates has vector field $\vec{T} = C\,r\,\hat{r}$, where C is a constant with correct units.

a- calculate $\vec{\nabla}\cdot\vec{T}$

b- Examine both sides of the divergences theorem for this vector field.

Ans:

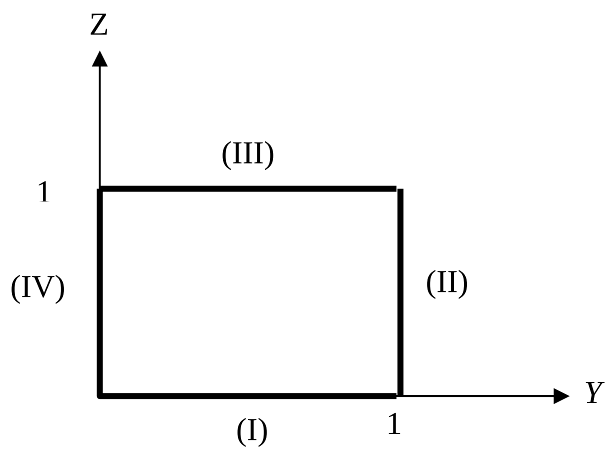
$$a- \quad \vec{\nabla}\cdot\vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Cr) = 3C$$

$$b-$$

$$I. \quad \oint_S \vec{T}\cdot d\,\vec{a} = \int_0^{2\pi}\int_0^{\pi} Cr(r^2\sin\theta d\theta d\varphi) = 4\pi C\, r^3\Big|_0^a = \underline{4\pi Ca^3}$$

$$II. \quad \int_V \left(\vec{\nabla}\cdot\vec{T}\right) d\tau = \int_0^{2\pi}\int_0^{\pi}\int_0^{a}(3C)(r^2\sin\theta dr d\theta d\varphi) = \underline{4\pi Ca^3}$$

3- For the electric field $\vec{E} = y^2\,\hat{\text{i}} + (2xy + z^2)\hat{\text{j}} + (3zy)\hat{\text{k}}$ calculate the integral $\oint \vec{E}.d\vec{l}$ for the square shown in the following figure.



Ans:

$$\nabla\times\vec{T} = \begin{vmatrix} \hat{\text{i}} & \hat{\text{j}} & \hat{\text{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy + z^2 & 3zy \end{vmatrix} = z\ \hat{\text{i}} .$$

$$\Rightarrow \oint \vec{T}.d\,\vec{l} = \oint \nabla\times\vec{T}.d\,\vec{a} = \int_0^1\int_0^1 z\ \hat{\text{i}}.(dxdz)\,\hat{\text{i}} = \frac{1}{2}.$$

It was a mistake that I used $\vec{E}$ rather than $\vec{T}$.

4- In the cylindrical coordinate, you are given the potential field $V = V_o \dfrac{s}{a}$, where V_o is a constant.

a- Calculate the electric field.

b- Find the energy stored in free space for the cylindrical region $0 < z < a$, $0 < \varphi < \pi, \quad 0 < s < 2$

Ans:

$$a - \quad \vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial s}\hat{s}\right] = -\frac{V_o}{a}\hat{s},$$

$$b - \quad W_E = \frac{\varepsilon_o}{2}\int_\tau \left|\vec{E}\right|^2 d\tau = \frac{\varepsilon_o}{2}\int_0^a\int_0^\pi\int_0^2 \left|\frac{V_o}{a}\right|^2 (s\ ds\ d\phi\ dz) = \frac{\pi\varepsilon_o V_o}{a}.$$

5- For a uniform charged sphere, of radius R and charge density ρ, find

a- the electric field at $r < R$ and at $r > R$.

b- the electric potential at $r < R$ and at $r > R$.

c- Plot the electric field and the electric potential versus r.

Ans: $\rho = \dfrac{Q}{\frac{4}{3}\pi r^3}$

a-

$$E = \frac{q_{in}}{r^2} = \begin{cases} k\,\dfrac{\rho(\frac{4}{3}\pi r^3)}{r^2} = k\,\dfrac{Q}{R^3}r & r < R \\ \\ k\,\dfrac{\rho(\frac{4}{3}\pi R^3)}{r^2} = k\,\dfrac{Q}{r^2} & r > R \end{cases}$$

b-

$$V = -\int_\infty^r \vec{E}.d\vec{l} = \begin{cases} k\,\dfrac{Q}{2R}(3 - \dfrac{r^2}{R^2}) & r < R \\ \\ k\,\dfrac{Q}{r} & r > R \end{cases}.$$

6- Given the potential field $V = \dfrac{\sin\theta}{r^2}$,

a- find $\vec{\nabla}^2 V$,

b- find the total charge stored inside the spherical shell $1 < r < 2$.

Ans:

a- $$\vec{\nabla}^2 V = \frac{2\sin\theta}{r^4} + \frac{\cos^2\theta - \sin^2\theta}{r^4 \sin\theta} = \frac{1}{r^4 \sin\theta}$$

b-

$$Q = \int \rho d\tau = -\varepsilon_o \int_1^2 \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^4 \sin\theta} r^2 \sin\theta d\theta d\phi dr$$

$$= -\varepsilon_o \left[-\frac{1}{r} \right]_1^2 (\theta)_0^{\pi} (\phi)_0^{2\pi} = -\varepsilon_o \left[-\frac{1}{2} + 1 \right] (\pi)(2\pi) = -\varepsilon_o \pi^2$$

7- Two infinite concentric cylindrical metal shell, with radii "a" and "b", $b > a$. The inner cylinder has a linear charge density "λ" and the outer cylinder has "$-\lambda$". Find

a- the electric field at s, $a < s < b$,

b- the potential difference between the two radii,

c- the capacitance of the system,

d- the energy stored in the system.

Ans:

a- Use Gauss' theorem

$$\phi = \int E.da = \frac{Q_{in}}{\varepsilon_o} \Rightarrow E(2\pi s l) = \frac{\lambda l}{\varepsilon_o} \Rightarrow E = \frac{2k\lambda}{s}, k = \frac{1}{4\pi\varepsilon_o}$$

b- $$V = \int_a^b E ds = 2k\lambda [\ln s]_a^b = 2k\lambda \ln\left(\frac{a}{b}\right)$$

c- $$C = Q/V = \lambda l / 2k\lambda \ln\left(\frac{b}{a}\right) = \frac{l}{2k \ln\left(\frac{b}{a}\right)}$$

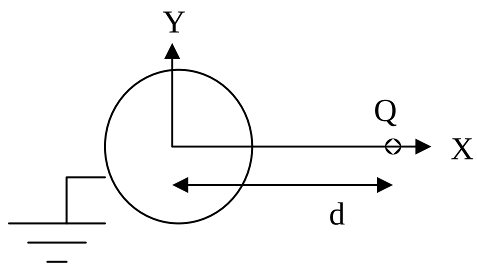
d- $$W = \frac{\varepsilon_o}{2} \int |E|^2 d^3 s = \frac{1}{8\pi} \int \left| \frac{2k\lambda}{s} \right|^2 s ds d\varphi dz =$$

$$= \lambda^2 k^2 l \ln\left(\frac{b}{a}\right) = \frac{1}{2} QV$$

8- In the following figure, a point charge "*Q*" is situated at a distance "$d = 2$ m" from the center of a grounded conducting sphere of a radius "$a = 0.5$ m".

b- calculate the value of the image charge and its position from the center of the sphere.

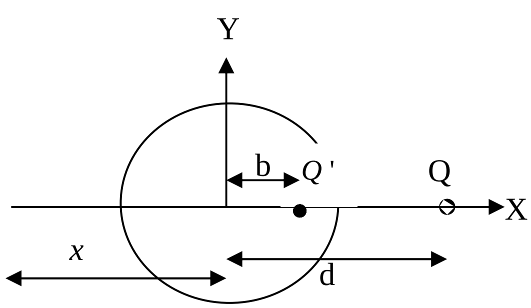
c- Find the point, not inside the sphere and not at infinity, where **E** is zero.



a-

b- Therefore, now all the electric field and potential can be obtained from the 2 point charges. Since Q and Q' lie on x-axis, the point where E = 0 (resultant field from the 2 charges) must be on x-axis. Further, since Q and Q' have opposite sign, that point cannot be in between Q and Q', because the field from Q and Q' both be in the same direction and will never cancel. Also, since point must lie to the left of Q'. Let the distance of this point on x-axis where $E = 0$ be *x*, then

$E = 0$



At point *x* on the negative of *x*-axis; $E = 0 \; \frac{q}{(d+x)^2} + \frac{q'}{(b+x)^2} \Rightarrow \frac{q}{(d+x)^2} - \frac{aq}{d(\frac{a^2}{d}+x)^2} = 0$

$\Rightarrow d(\frac{a^2}{d}+x)^2 = a(d+x)^2$ Or

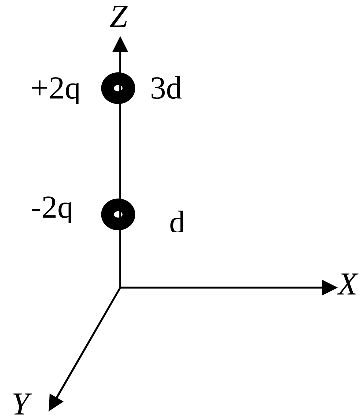
$$(\frac{1}{8}+x) = \pm\sqrt{\frac{1}{4}}(2+x)$$

$$(\frac{1}{8}+x) = \pm(1+\frac{x}{2}) \Rightarrow \frac{x}{2} = \pm 1 - \frac{1}{8}.$$

9- In the figure (The *XY* plane is a grounded conductor.)

d- Find the images of the charges and their corresponding positions.

e- Find the force on the charge $+2q$.



Ans:

a- We will have 2 images: ($-2q$ at –3d), and ($+2q$ at –d),

b- The force on the charge $+2q$ will be:

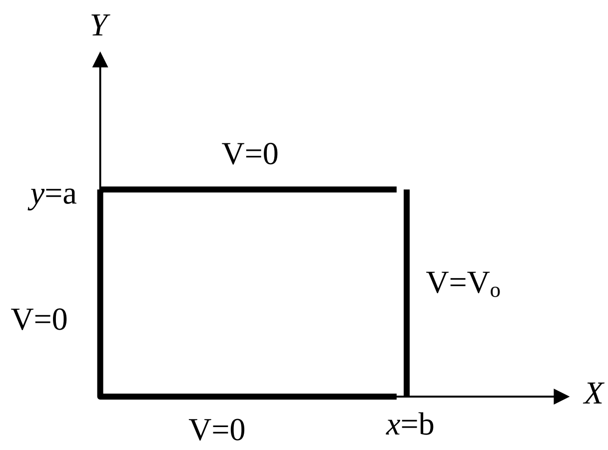
$$\vec{F} = k\,(2q)\left[-\frac{2q}{(2d)^2}+\frac{2q}{(4d)^2}-\frac{2q}{(6d)^2}\right]\hat{Z}$$
$$= k\,\frac{4q^2}{d^2}\left[-\frac{1}{4}+\frac{1}{16}-\frac{1}{36}\right]\hat{Z} = \underline{-\frac{31}{36}\frac{kq^2}{d^2}\hat{Z}}$$

10- The figure shows a rectangular box with different potential on its sides.

a- Write down the boundary conditions,

b- Write down the general solution by using the above boundary conditions.

c- Use Fourier's trick to calculate the final form of the potential.



[Hint:

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right) = \delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ \frac{a}{2}, & \text{if } m = n \end{cases},$$

$$\int_0^a \sin\left(\frac{n\pi y}{a}\right)dy = \begin{cases} 0 & \text{if n is even} \\ \frac{2a}{n\pi} & \text{if n is odd} \end{cases}$$]

Ans:

a- The boundary conditions are

$$V(x,0)=0,\quad V(x,a)=0,\quad V(0,y)=0,\quad V(b,y)=V_o$$

b- $\alpha_n=\beta_n$. Use the boundary condition

$$V(0,y=a)=0\Rightarrow \sin\alpha a=0\Rightarrow \alpha_n=\frac{n\pi}{a},\qquad n=1,2,3,\cdots$$

Then the general solution is $V(x,y)=\sum_{n=1}^{\infty}C_n\sinh(\alpha_n b)\sin(\beta_n y)$,

c- Use the Fourier's trick to calculate

$$C_n\sinh(\frac{n\pi b}{a})=\frac{2}{a}\int_0^a V_o\sin(\frac{n\pi y}{a})\,dy,$$

$$C_n=\frac{2V_o}{a\sinh(\frac{n\pi b}{a})}\int_0^a\sin(\frac{n\pi y}{a})dy=\frac{2V_o}{a\sinh(\frac{n\pi b}{a})}\times\left\{\begin{array}{ll}0 & \text{if n is even}\\ \frac{2a}{n\pi} & \text{if n is odd}\end{array}\right\}.$$

Finally

$$V(x,y)=\frac{4V_o}{\pi}\sum_{n=1,3,5,\cdots}^{\infty}\frac{\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})}{n\sinh(\frac{n\pi b}{a})}$$