<u>Example 3.3</u> Using the separation of variables to solve the Lap lace's equation:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0 \qquad (1)$$

With the following boundary conditions:

$$V(x,0)=0 \qquad (i)$$
$$V(x,a)=0 \qquad (ii)$$
$$V(0,y)=V_o \qquad (iii)$$
$$V(\infty,y)=0 \qquad (iv)$$

First: in equation (1) use the product:

$$V(x,y)=X(x)Y(y) \qquad (A)$$

You can find:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0 \quad \Rightarrow \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2}=C_1, \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=C_2$$

where $C_1+C_2=0$. Which one will be positive? And which one will be negative?

Second: Take $C_1=k^2$?, we reach to the equations:

$$\frac{d^2X}{dx^2}=k^2X\,, \qquad (2)$$
$$\frac{d^2Y}{dy^2}=-k^2Y\,. \qquad (3)$$

Hint:

The solution of equation (2) could have the following expressions:

$$X(x)=Ae^{kx}+Be^{-kx} \qquad (2.a)$$

which represent a decaying (or increasing) function, or

$$X(x)=C\cosh(kx)+D\sinh(kx) \qquad (2.b)$$

The solution of equation (3) could have the following expressions:

$$Y(y)=Ae^{iky}+Be^{-iky} \qquad (3.a)$$

or

$$Y(y)=C\cos(ky)+D\sin(ky) \qquad (3.b)$$

Then equation (A) will take the form:

$$V(x,y)=X(x)Y(y)=\left(Ae^{kx}+Be^{-kx}\right)\left(C\cos(ky)+D\sin(ky)\right) \qquad (B)$$

Third: Use the boundary conditions:

$$\text{B.C. (iv)} \Rightarrow A = 0$$
$$\text{B.C. (i)} \Rightarrow C = 0$$

So

$$V(x,y) = X(x)Y(y) = N\, e^{-kx} \sin(ky) \qquad \text{(C)}$$

where N is a new constant and will be determined later using the boundary condition (iii).

$$\text{B.C. (ii)} \Rightarrow \sin(ka) = 0 \Rightarrow k = \frac{n\pi}{a}, \qquad (n = 1, 2, 3, \cdots)$$

$n = 0$ is not good because the potential will be vanished everywhere. The positive values of n is taken because we are looking for a solution in the first quadrant.

From the superposition principle, equation (C) will be:

$$V(x,y) = \sum_{n=1}^{\infty} N_n\, e^{-n\pi x/a} \sin(\frac{n\pi y}{a}) \qquad \text{(D)}$$

Boundary condition (iii) implies that:

$$V(0,y) = \sum_{n=1}^{\infty} N_n \sin(\frac{n\pi y}{a}) = V_o(y) \qquad \text{(E)}$$

To calculate we will use Fourier's trick, by multiplying equation (E) by $\sin(\frac{m\pi y}{a})$ (where m is a positive integer) and integrate from 0 to a.

$$\sum_{n=1}^{\infty} N_n \int_0^a \sin(\frac{n\pi y}{a}) \sin(\frac{m\pi y}{a}) dy = \int_0^a V_o(y) \sin(\frac{m\pi y}{a}) dy,$$

Use the standard integral:

$$\int_0^a \sin(\frac{n\pi y}{a}) \sin(\frac{m\pi y}{a}) = \delta_{mn} = \begin{cases} 0, & \text{if } m \neq n \\ \frac{a}{2}, & \text{if } m = n \end{cases}$$

then

$$N_n = \frac{2}{a} \int_0^a V_o(y) \sin(\frac{m\pi y}{a}) dy,$$

If $V_o(y)$ is constant, one finds

$$N_n = \frac{2V_o}{a} \int_0^a \sin(\frac{m\pi y}{a}) dy = \frac{2V_o}{m\pi}(1 - \cos m\pi) = \begin{cases} 0, & \text{if } m \text{ is even,} \\ \frac{4V_o}{m\pi}, & \text{if } m \text{ is odd} \end{cases}$$

Since m and n are dummy indices, finally

$$\boxed{V(x,y) = \frac{4V_o}{\pi} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n} e^{-n\pi x/a} \sin(\frac{n\pi y}{a})}$$