**KING FAHD UNIVERSITY OF PETROLUIM AND MINERALS**
**DEPARTMENT OF PHYSICS**

# ELECTRICITY AND MAGNETISM I

## PHYS 305

## SPRING 2008

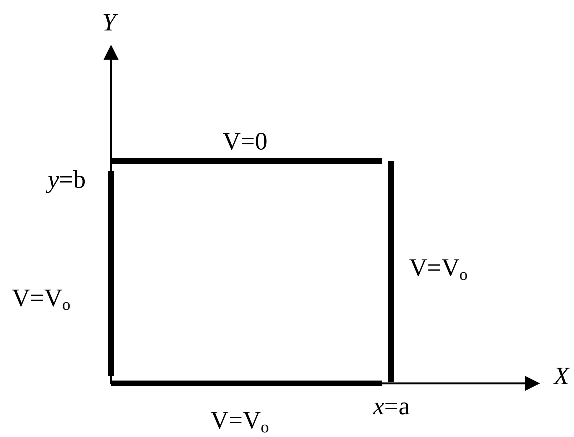
## HOME WORK # 5

## DUE DATE (19/4/2008)

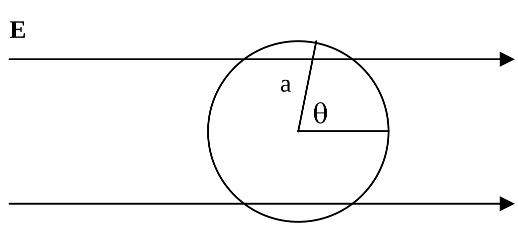
PROF. I. NASSER

SOLVE THE FOLLOWING PROBLEMS.

1- A duct is infinitely long in the *z* direction and has a rectangular cross section bounded by the planes $x = 0$, $x = a$, $y = 0$, and $y = b$. The plane $y = b$ is at zero potential, and the other planes are at potential $V_o$. With two different methods, find the potential everywhere inside the duct. (solve the problem using two different ways)

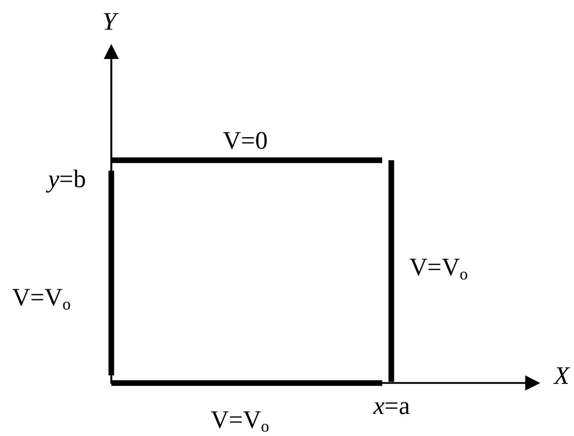



2- A conducting sphere of radius "a" bearing total charge "Q" is placed in an initially uniform electric field $\vec{E} = E_o\,\hat{z}$. Find the potential at all points exterior to the sphere.



## Answer:

1- A duct is infinitely long in the $z$ direction and has a rectangular cross section bounded by the planes $x$ = 0, $x$ = $a$, $y$ = 0, and $y$ = $b$. The plane $y$ = $b$ is at zero potential, and the other planes are at potential $V_o$. With two different methods, find the potential everywhere inside the duct.




**Answer: B.C. required:**

$V(0,y) = V_o$ (i)

$V(a,y) = V_o$ (ii)

$V(x,0) = V_o$ (iii)

$V(x,b) = 0$ (iv)

Try the trigonometric function in x-direction and the exponential in y-direction.

$$V(x,y) = Ax + By + C + \sum_k^{\infty} \left[a_k \cos(kx) + b_k \sin(kx)\right]\left[c_k \cosh(ky) + d_k \sinh(ky)\right]$$

**B.C. (i) required that:**

$$V(0,y) \equiv V_o = 0 + By + C + \sum_k^{\infty} a_k \left[c_k \cosh(ky) + d_k \sinh(ky)\right]$$

which implies:

$$B = 0; \quad C = V_0; \quad \text{and} \quad a_k = 0$$

$$\therefore\quad V(x,y) = Ax + V_o + \sum_k^{\infty} \sin(kx)\left[c'_k \cosh(ky) + d'_k \sinh(ky)\right]$$

**B.C. (ii) required that:**

$$V(a,y) \equiv V_o = Aa + V_o + \sum_k^{\infty} \sin(ka)\left[c'_k \cosh(ky) + d'_k \sinh(ky)\right]$$

which implies:

$$A = 0; \quad \text{and} \quad ka = n\pi, \qquad n = 1,2,3,\cdots$$

Therefore $V(x,y)$ reduces to:

$$V(x,y) = V_o + \sum_{n=1,2,}^{\infty} \sin(\frac{n\pi}{a}x)\left[c'_k \cosh(\frac{n\pi}{a}y) + d'_k \sinh(\frac{n\pi}{a}y)\right]$$

**B.C. (iii) required that:**

$$V(x,0) \equiv V_o = V_o + \sum_{n=1,2,}^{\infty} \sin(\frac{n\pi}{a}x)\left[c'_k\right] \quad \Rightarrow \quad c'_k = 0$$

then

$$V(x,y) = V_o + \sum_{n=1,2,}^{\infty} d'_k \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}y)$$

**B.C. (iv) required that:**

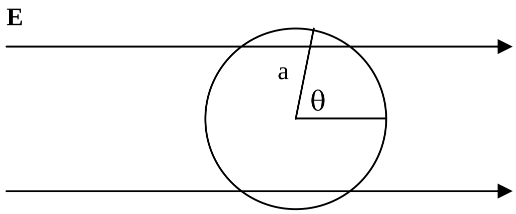
$$-V_o = \sum_{n=1,2,}^{\infty} d'_k \sin(\frac{n\pi}{a}x)\sinh(\frac{n\pi}{a}b)$$

To calculate the constant $d'_k$, we have to use the Fourier's trick. Multiply the above equation with $\sin(\frac{m\pi}{a}x)$ and integrate over $x$:

$$\underbrace{-V_o\int_0^a \sin(\frac{m\pi}{a}x)dx}_{\frac{V_o a}{m\pi}\left[1-(-1)^m\right]} = \sum_{n=1,2,}^{\infty} d'_k \sinh(\frac{n\pi}{a}b)\underbrace{\int_0^a \sin(\frac{m\pi}{a}x)\sin(\frac{n\pi}{a}x)dx}_{=\frac{a}{2}\delta_{mn}}$$

$$\therefore V(x,y) = V_o - \frac{4V_o}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin(\frac{n\pi}{a}x)\frac{\sinh(\frac{n\pi}{a}y)}{\sinh(\frac{n\pi}{a}b)}$$

2- A conducting sphere of radius "a" bearing total charge "Q" is placed in an initially uniform electric field $\vec{E} = E_o\,\hat{z}$. Find the potential at all points exterior to the sphere.




Ans: Start with the expression for the potential in the form:

$$V\,(r,\theta) = \frac{Q}{r} + \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)P_n(\cos\theta)$$

Now, using the B. C, as $r \to \infty$ we wish the potential to behave as $-E_o z = -Er\cos\theta$

Thus $A_1 = -E_o$ and all others $A_n$'s are zero.

$$V(r,\theta) = \frac{Q}{r} - E_o r\cos\theta + \sum_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos\theta)$$

Now, we have to make sure that this potential satisfy the B.C. at $r = a$,

$$V\,(a,\theta) = \frac{Q}{a} = \frac{Q}{a} - E_o a\cos\theta + \sum_{n=0}^{\infty} B_n a^{-n-1} P_n(\cos\theta)$$

$$= -E_o a\cos\theta + \frac{B_o}{a} + \frac{B_1}{a^2}P_1(\cos\theta) + \frac{B_2}{a^3}P_2(\cos\theta) + \cdots$$

$$\Rightarrow B_o = 0, \quad B_1 = E_o a^3, \quad B_n = 0 \quad \text{for } n \neq 0,1.$$

Thus the final solution that satisfy all B. C. is

$$V(r,\theta) = \left(\frac{a^3}{r^3} - 1\right)E_o r\cos\theta + \frac{Q}{r}.$$