Chapter 9
Center of Mass and Linear Momentum

PREVIEW

The center of mass plays a major role in using Newton's law for an object that consists of a huge number of particles. The *momentum* of an object is the product of its mass and velocity. If you want to change the momentum of an object, you must apply an *impulse*, which is the product of force and the time during which the force acts. If there are no external forces acting on a system of objects, the momentum is said to be conserved, that is, the total momentum of the system before some event (like a collision) is equal to the total momentum after that event. In this chapter, we will discuss examples of both one- and two-dimensional collisions.

QUICK REFERENCE

Important Terms (not in alphabetic order)

Center of mass

The point at which the total mass of a system of masses can be considered to be concentrated.

Linear momentum (p = *m*v)

The product of the mass of an object and its velocity. Momentum is a vector quantity, and thus the total linear momentum of a system of objects is the vector sum of the individual momenta of the objects in the system.

Equations and Symbols (For the whole chapter)

$x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ $\mathbf{p} = m\mathbf{v}$ $\Delta\mathbf{p} = m(\mathbf{v}_f - \mathbf{v}_0)$	$\mathbf{p}$ = momentum m = mass $\mathbf{v}$ = velocity x_{cm} = position of the center of mass of a system of particles x_1 = position of a mass relative to a chosen origin

Basic Requirements:

1. Understand the concept of the center of mass
2. Master Newton's second law for a system of particles
3. Understand the concept of linear momentum

9-1 CENTER OF MASS

For a system of particles (that is, lots of them) there is a special point in space known as the **center of mass** which is of great importance in describing the overall motion of the system. This point is a weighted average of the positions of all the mass points.

Center of mass is:

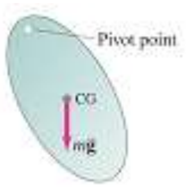
"The point at which the total mass of a system of masses can be considered to be concentrated".

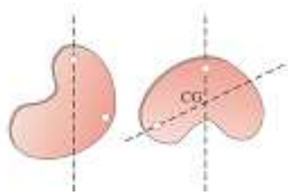
Center of mass principles:

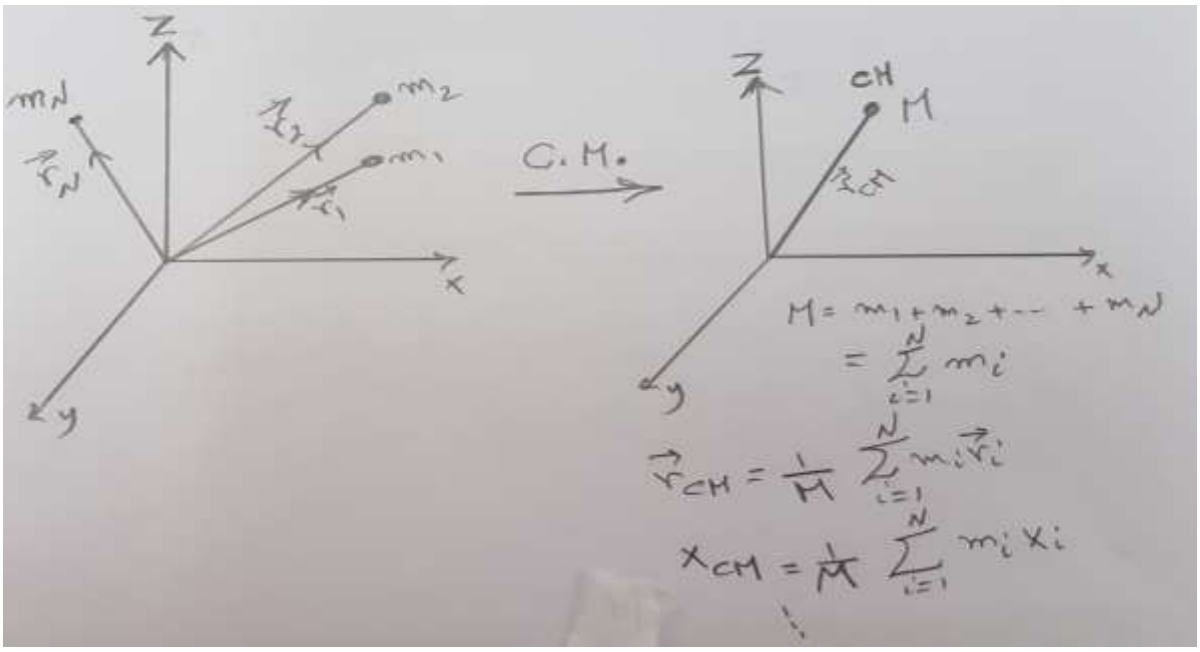
1. A system moves as if all of its mass were concentrated at the center of the system.
2. The forces acting on a system act as if directed toward the center of mass of the system.

The **center of gravity** is the point where the gravitational force can be considered to act. It is the same as the center of mass as long as the gravitational force does not vary among different parts of the object.

The center of gravity can be found experimentally by suspending an object from different points. The CM need not be within the actual object – a doughnut's CM is in the center of the hole.







If the particles in the system have masses $m_1, m_2, \cdots, m_N$, with total mass

$$M = \sum_{i=1}^{N} m_i = m_1 + m_2 + \cdots + m_N$$

and respective positions $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$, then the center of mass $\boldsymbol{r}_{\text{CM}}$ is:

$$\boldsymbol{r}_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{N} m_i \boldsymbol{r}_{\text{i}}$$

which means that the *x*, *y* and *z* coordinates of the center of mass are

$$x_{\mathrm{CM}} = \frac{1}{M}\sum_{i=1}^{N} m_i x_i, \quad y_{\mathrm{CM}} = \frac{1}{M}\sum_{i=1}^{N} m_i y_i, \quad z_{\mathrm{CM}} = \frac{1}{M}\sum_{i=1}^{N} m_i z_i$$

For an extended object (i.e. a continuous distribution of mass) the definition of $\boldsymbol{r}_{\mathrm{CM}}$ is given by an integral over the mass elements of the object (not needed in our course):

$$\boldsymbol{r}_{\mathrm{CM}} = \frac{1}{M}\int \boldsymbol{r}\, dm$$

which means that the *x*, *y* and *z* coordinates of the center of mass are now:

$$x_{\mathrm{CM}} = \frac{1}{M}\int x\,dm, \quad y_{\mathrm{CM}} = \frac{1}{M}\int y\,dm, \quad z_{\mathrm{CM}} = \frac{1}{M}\int z\,dm, \qquad (9.9)$$

If the density (mass per unit volume) is uniform as,

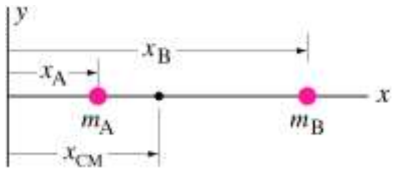
$$\rho = \frac{dm}{dV} = \frac{M}{V}, \qquad (9\text{-}10)$$

then Eq.9-9 can be written as

$$x_{com} = \frac{1}{V}\int x\,dV \quad , \quad y_{com} = \frac{1}{V}\int y\,dV \quad , \quad z_{com} = \frac{1}{V}\int z\,dV \qquad (9\text{-}11)$$

where *V* is the volume occupied by *M*.

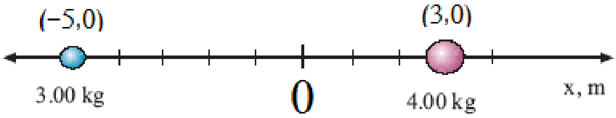
Example: For two particles, with masses m_a and m_b, the center of mass lies closer to the one with the most mass:



$$x_{\text{CM}} = \frac{m_A x_A + m_B x_B}{m_A + m_B}, \qquad M = m_A + m_B$$

where M is the total mass.

Example: A 3.00 kg particle is located on the x axis at $x = -5.00$ m and a 4.00 kg particle is on the x axis at $x = 3.00$ m. Find the center of mass of this two–particle system.

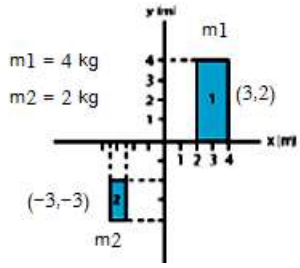


Answer: The masses are shown in the figure. There is only one coordinate (x) and two mass points to consider here; using the definition of x_{CM}, we find:

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{(3.00\text{ kg})(-5.00\text{ m}) + (4.00\text{ kg})(3.00\text{ m})}{(3.00\text{ kg} + 4.00\text{ kg})}$$
$$= -0.429\text{ m.}$$

The center of mass is located at $x = -0.429$ m.

Example: The location of two thin flat objects of masses $m_1 = 4.0$ kg and $m_2 = 2.0$ kg are shown in the figure, where the units are in m. The x and y coordinates of the center of mass of this system are:



Answer: The coordinates of the CM of flat 1 is (3, 2). The coordinates of the CM of flat 2 is (-3,-3).

$$x_{\text{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{4.0 \times 3.0 + 2.0 \times (-3.0)}{6.0} = \underline{1.0\text{ m}}$$

$$y_{\text{CM}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{4.0 \times 2.0 + 2.0 \times (-3.0)}{6.0} = \underline{0.33\text{ m}}$$

9-2 Newton's Second Law for a System of Particles

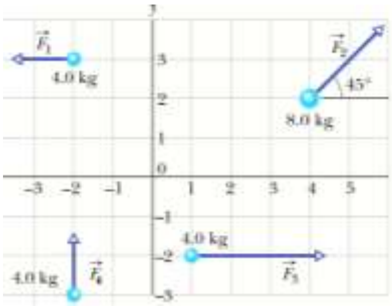
The motion of the center mass of any system of particles is governed by **Newton's second law** for a system of particles, which is

$$\boldsymbol{F}_{\text{net}} = M\,\boldsymbol{a}_{\text{com}} \qquad (9\text{-}14)$$

Here $\vec{F}_{net}$ is the net force of all the *external* force acting on the system. Internal forces are not included. M is the total mass of the system; should be a constant, i.e. the system is **closed**.
$\boldsymbol{a}_{\text{com}}$ is the acceleration of the system's center of mass.
In components:

$$\boldsymbol{F}_{\text{net},x} = M\,\boldsymbol{a}_{\text{com},x},\quad \boldsymbol{F}_{\text{net},y} = M\,\boldsymbol{a}_{\text{com,y}},\quad \boldsymbol{F}_{\text{net,z}} = M\,\boldsymbol{a}_{\text{com,z}}$$

Example: The four particles in Figure are initially at rest. Each experiences an external force. The directions are indicated, and the magnitudes are F_1 = 6.0 N, F_2 = 12 N, F_3 = 14 N, and F_4 = 6 N. In what direction θ does the center of mass move?



Solution: the given information are given in the following table:

n	F_n(N)	Mass (kg)	(x-axis, y-axis) m
1	6.0 (- **i** +0 **j**)	4.0	(-2, 3)
2	12 ($\cos 45^o$ **i** + $\sin 45^o$ **j**)	8.0	(4, 2)
3	14 (1 **i** +0 **j**)	4.0	(1,-2)
4	6 (0 **i** +1 **j**)	4.0	(-2,-3)

H.W. Calculate the coordinate of the center of mass.
The acceleration of the CEM will be:

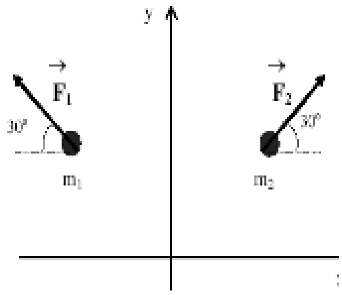
$$a_{com,x} = \frac{1}{M}\sum_{i=1}^{4} F_{ix} = \frac{F_{1x}+F_{2x}+F_{3x}+F_{4x}}{M_1+M_2+M_3+M_4} = \frac{-6.0\text{ N} + (12\text{ N})\cos 45^o + 14\text{ N} + 0\text{ N}}{20\text{ kg}} = 0.824\ \text{m/s}^2,$$

$$a_{com,y} = \frac{1}{M}\sum_{i=1}^{4} F_{iy} = \frac{F_{1y}+F_{2y}+F_{3y}+F_{4y}}{M_1+M_2+M_3+M_4} = \frac{0\text{ N} + (12\text{ N})\sin 45^o + 0\text{ N} + 6.0\text{ N}}{20\text{ kg}} = 0.724\ \text{m/s}^2$$

The direction θ does the center of mass move is:

$$\theta = \tan^{-1}(a_{com,y}/a_{com,x}) = 41.3^o$$

Example: Two particles m_1 and m_2, 5.0-kg each, are initially at rest. External forces F_1 and F_2, 12 N each, are acting on these particles as shown in Figure. The acceleration of the center of mass of the two particles system is:



Answer:

$$\because (m_1+m_2)\vec{a}_{cm} = m_1\vec{a}_1 + m_2\vec{a}_2 = \vec{F}_1 + \vec{F}_2$$

$$\therefore \vec{a}_{cm} = \frac{1}{M}\sum_{i=1}^{2}\vec{F}_i$$

$$\vec{a}_{cm} = \frac{1}{M}\sum_{i=1}^{2}\vec{F}_i = \frac{1}{10}\left(\vec{F}_1+\vec{F}_2\right), \qquad m_1 = m_2 = m = 5\text{ kg}$$

$$= 0.1\left(-\left|\vec{F}_1\right|\cos 30^\circ\ \hat{\text{i}} + \left|\vec{F}_1\right|\sin 30^\circ\ \hat{\text{j}}\right) + \left(\left|\vec{F}_2\right|\cos 30^\circ\ \hat{\text{i}} + \left|\vec{F}_2\right|\sin 30^\circ\ \hat{\text{j}}\right), \qquad \left|\vec{F}_2\right| = \left|\vec{F}_1\right| = 12$$

$$= 0.1\left(0.0\ \hat{\text{i}} + 2\left|\vec{F}_2\right|\sin 30^\circ\ \hat{\text{j}}\right) = \underline{\left(0.0\ \hat{\text{i}} + 1.2\ \hat{\text{j}}\right)\text{m/s}^2}$$

9-3 Linear Momentum

The **linear momentum "P"** of a particle with mass m moving with velocity **v** is defined as:

$$\mathbf{P} = m\,\mathbf{v}$$

Linear momentum is a vector. Direction of **p** = direction of velocity **v.** When giving the linear momentum of a particle you must specify its magnitude and direction. The units of $[\mathbf{P}] = \text{kg.m/s}$ (no special name).

Definition: Total momentum of several masses: m_1 with velocity $\mathbf{v}_1$, m_2 with velocity $\mathbf{v}_2$, etc..

$$\vec{p}_{tot} \equiv \sum_i \vec{p}_i = \vec{p}_1 + \vec{p}_2 + \ldots = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots$$

- Linear momentum is an extremely *useful* concept because total momentum is *conserved* in a system **isolated** from outside forces. Momentum is especially useful for analyzing collisions between particles.
- Another way of writing the linear momentum of a system of particles is:

$$\mathbf{P} = M\,\boldsymbol{v}_{\text{CM}}$$

"The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass".

- The momentum of a particle is related to the net force on that particle in a simple way; since the mass of a particle remains constant, if we take the time derivative of a particle's momentum we find

$$\frac{d\mathbf{P}}{dt} = m\frac{d\mathbf{v}}{dt} = m\,\boldsymbol{a} = \boldsymbol{F}_{net} \qquad \Rightarrow \quad \boldsymbol{F}_{net} = \frac{d\mathbf{P}}{dt}$$

Example: A 3.00 kg particle has a velocity of (3.0 **i** − 4.0 **j**) m/s. Find its x and y components of momentum and the magnitude of its total momentum.

Answer: Using the definition of momentum and the given values of m and **v** we have:

$$\mathbf{p} = m\mathbf{v} = (3.00\text{ kg})(3.0\,\mathbf{i} - 4.0\,\mathbf{j}) = (9.0\,\mathbf{i} - 12.\,\mathbf{j})$$

So the particle has momentum components

$$p_x = +9.0\text{ kg.m/s} \qquad \text{and} \qquad p_y = -12\text{ kg.m/s}$$

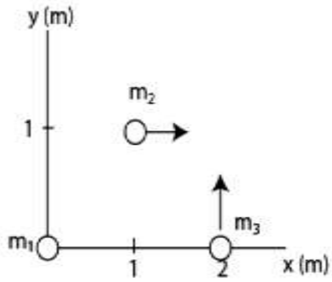
H.W. Find the magnitude and the direction of the momentum.

Example: Each object in the figure has a mass of 2.0 kg. The mass m_1 is at rest, m_2 has a speed of 3.0 m/s in the direction of +ve x-axis and m_3 has a speed of 6.0 m/s in the direction of +ve y-axis. Calculate

i- The center of mass of the system,

ii- The momentum of the center of mass of the system

iii- The velocity of the center of mass



Answer: $m_1 = m_2 = m_3 = 2.0$ kg. Coordinates of: $m_1 = (0, 0)$, $m_2 = (1, 1)$, $m_3 = (2, 0)$

$$x_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{2.0\times(0.0)+2.0\times(1.0)+2.0\times(2.0)}{2.0+2.0+2.0} = \underline{1.0\text{ m}}$$

$$y_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{3} m_i y_i = \frac{2.0\times(0.0)+2.0\times(1.0)+2.0\times(0.0)}{2.0+2.0+2.0} = \underline{0.33\text{ m}}$$

i- $\mathbf{P} = \sum_{i=1}^{3} m_i\,\boldsymbol{v}_i = m_1(0) + 2(3\,\hat{i}) + 2(6\,\hat{j}) = \left(6.0\,\hat{i} + 12\,\hat{j}\right)\frac{\text{kg.m}}{\text{s}}$

ii- $\mathbf{P} = M\,\boldsymbol{v}_{\text{CM}} \quad \Rightarrow \quad \boldsymbol{v}_{\text{CM}} = \frac{\mathbf{P}}{M} = \left(1.0\,\hat{i} + 2.0\,\hat{j}\right)$ m/s

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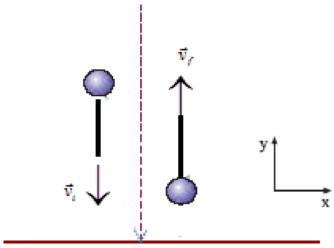


Example: A 1.0 kg ball falling vertically hits a floor with a velocity of 3.0 m/s and bounces vertically up with a velocity of 2.0 m/s. If the ball is in contact with the floor for 0.1 s, what is the change in the linear momentum of the ball?

Answer: Define the coordinates as given in the figure, so $\vec{v}_i = -3.0\ \hat{j}$ and $\vec{v}_f = 2.0\ \hat{j}$. The change of linear momentum of the ball is given by:

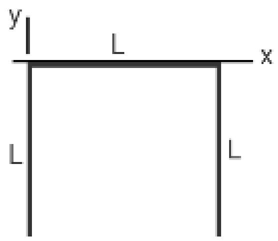
The change of linear momentum of the ball is given by:

$$\Delta \mathbf{P} = m\,\Delta\vec{v} = m\left(\vec{v}_f - \vec{v}_i\right) = (1)\left[2.0\ \hat{j} - (-3.0\ \hat{j})\right] = 5.0\ \hat{j}$$

Note: In this type of problem be very careful of the **signs**.

Extra Problems

Q: Three uniform thin rods, each of length L = 20 cm, form an inverted U shape as shown in **Figure**. Each one of the vertical rods has a mass of 20 g and the horizontal rod has a mass of 60 g. What is the x and **y** coordinates of the center of mass of the system, respectively?



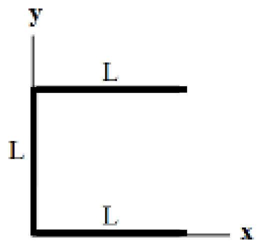
Answer:

Horizontal rod: m1 = 60 g, (x,y) = (L/2,0) = (10,0)
Left rod: m2 = 20 g, (x,y) = (0,-L/2) = (0,-10)
Right rod: m3 = 20 g, (x,y) = (L,-L/2) = (20,-10)
M=m1+m2+m3=60+20+20

$$x_{CM} = \frac{1}{M}\sum_{i=1}^{3} m_i x_i = \frac{60(10)+20(0)+20(20)}{100} = +10 \text{ cm},$$

$$y_{CM} = \frac{1}{M}\sum_{i=1}^{3} m_i y_i = \frac{60(0)+20(-10)+20(-10)}{100} = -4.0 \text{ cm}.$$

Q: An object is formed by three identical uniform thin rods, each of length L and mass M, as shown in **Figure**. Determine the x and y coordinates, (x, y), of the center of mass of this object.

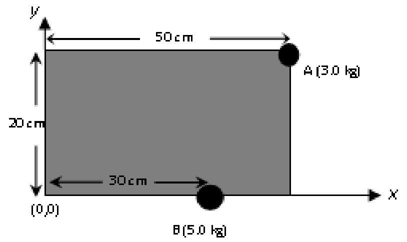


Ans:

$$x_{cm} = \frac{M(0)+M(L/2)+M(L/2)}{3M} = L/3$$

$$y_{cm} = \frac{M(L/2)+M(0)+M(L)}{3M} = L/2$$

Q: A uniform and thin rectangular piece of wood of width 20 cm and length 50 cm has a mass of 2.0 kg. Two point masses 3.0 kg and 5.0 kg are attached to it at points A and B, respectively (see **Figure**). Find the *x* and *y* coordinates, respectively, of the center of mass of the system relative to the origin.

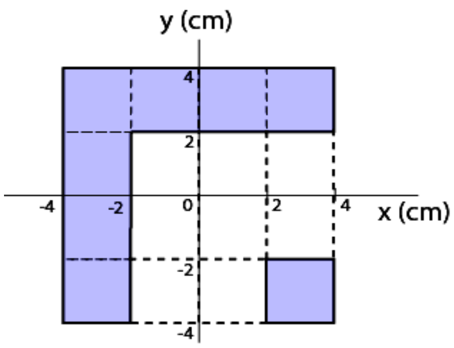


Ans:

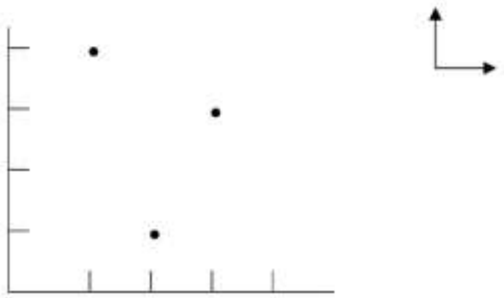
$$x_{com} = \frac{(5)(30)+(3)(50)+(2)(25)}{10} = 35 \text{ cm}$$

$$y_{com} = \frac{(2)(10)+(3)(20)}{10} = 8 \text{ cm}$$

Q: The two pieces of uniform sheets made of the same metal are placed in the x-y plane as shown in the figure. The center of mass (x_{com}, y_{com}) of this arrangement is: Ans: (-0.75, 0.75) cm



Example 1. Find the center of mass for a three body system, m_1, m_2, m_3 = 3kg, as shown below. The tick marks are 1 meter increments.



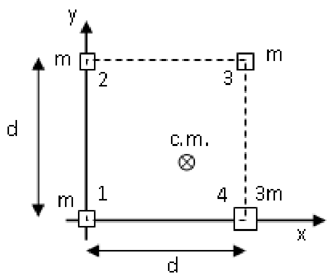
$$x_{cm} = \frac{x_1 m_1 + x_2 m_2 + x_3 m_3}{m_1 + m_2 + m_3} = \frac{(1m)(3kg) + (2m)(3kg) + (3m)(3kg)}{3kg + 3kg + 3kg} = \frac{18m \cdot kg}{9kg} = 2.0m$$

$$y_{cm} = \frac{y_1 m_1 + y_2 m_2 + y_3 m_3}{m_1 + m_2 + m_3} = \frac{(1m)(3kg) + (3m)(3kg) + (4m)(3kg)}{3kg + 3kg + 3kg} = \frac{24m \cdot kg}{9kg} = 2.7m$$

$(x, y)_{cm}$ = (2.0, 2.7) meters

Q: Where is c.m. of this 4 mass system? The masses, labeled 1, 2, 3, 4, form a square of edge length d. The four masses are m, m, m, and 3m.

Answer:

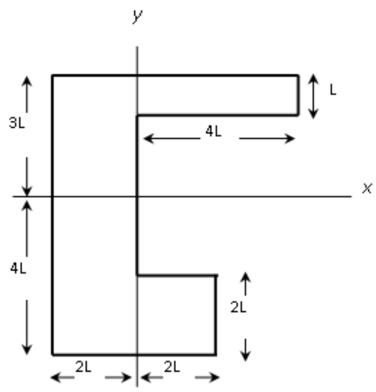


$$X = \frac{1}{M}\left(m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4\right)$$
$$= \frac{1}{6m}\left(m \cdot 0 + m \cdot 0 + m \cdot d + 3m \cdot d\right) = \frac{4}{6} d \approx 0.66 d$$
$$Y = \frac{1}{M}\left(m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4\right)$$
$$= \frac{1}{6m}\left(m \cdot 0 + m \cdot d + m d + 3m \cdot 0\right) = \frac{1}{3} d \approx 0.33 d$$

Notice that the c.m. is closer to the heavy corner in the lower right. Roughly speaking, the c.m. is the "balance point".

Q: What are (a) the *x* coordinate and *y* coordinate of the center of mass for the uniform plate shown in the Figure if L = 5.0 cm?

Solution:

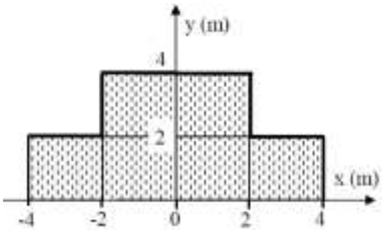


Since the plate is uniform, we can split it up into three rectangular pieces, with the mass of each piece being proportional to its area and its center of mass being at its geometric center. We'll refer to the large 35 cm × 10 cm piece (shown to the left of the *y* axis in the Figure) as section 1; it has 63.6% of the total area and its center of mass is at (x_1 ,y_1) = (-5.0 cm, -2.5 cm). The top 20 cm × 5 cm piece (section 2, in the first quadrant) has 18.2% of the total area; its center of mass is at (x_2,y_2) = (10 cm, 12.5 cm). The bottom 10 cm ×10 cm piece (section 3) also has 18.2% of the total area; its center of mass is at (x_3,y_3) = (5 cm, -□15 cm).

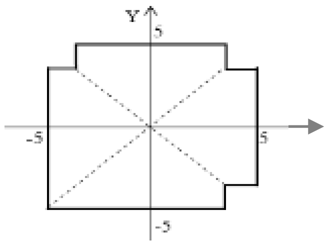
(a) The x coordinate of the center of mass for the plate $x_{com} = (0.636)x_1 + (0.182)x_2 + (0.182)x_3 = -0.45$ cm .

(b) The y coordinate of the center of mass for the plate $y_{com} = (0.636)y_1 + (0.182)y_2 + (0.182)y_3 = -2.0$ cm .

Q: A uniform plate is shaped as in the Figure. Find the center of mass of the plate? (0, 1.7) m

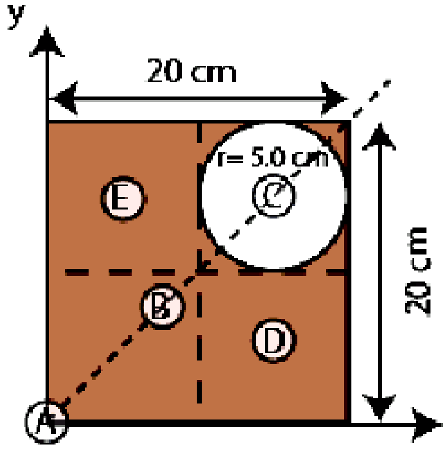


Q: The **Figure** shows a uniform square sheet from which three identical corners are removed. Calculate the location of its center of mass.



a. in the third quadrant. b. along the x-axis
c. along the y-axis d. in the first quadrant.
e. in the second quadrant.

Q: A circular hole of radius 5.0 cm is cut from a uniform square of metal sheet having sides 20 cm as shown in the figure. Calculate the location of its center of mass. Which point could be the center of mass of this sheet? (**Ans:** Point B)



Answer: use ρ to define the surface density of the metal sheet, then

$M_1 = (20\times 20)\rho = 400\rho$ kg, and its coordinate at (10 cm,10 cm),

$M_2 = -(\pi\times 5^2)\rho = -78.5\rho$ kg, and its coordinate at (15 cm,15 cm).

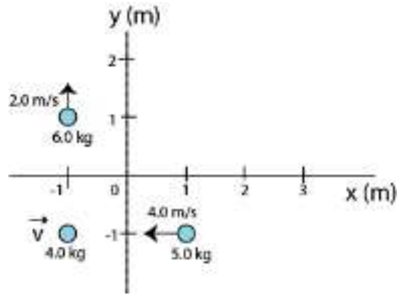
The minus sign for the mass M_2 because it is a missing part from the sheet.

The total mass of the system will be $M = M_1 - M_2 = 321.5\rho$ kg, and its coordinate will be:

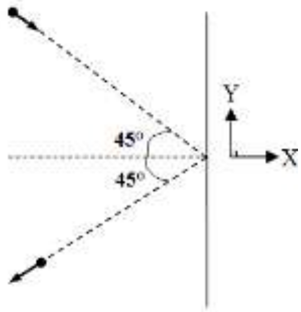
$$x_{CM} = \left[\frac{M_1\times 10 + (-M_2)\times 15}{M}\right] = \left[\frac{4000\rho - 1178\rho}{321.5\rho}\right] = \frac{2833}{321.5} = 8.8 \text{ cm}$$

$$y_{CM} = \left[\frac{M_1\times 10 + (-M_2)\times 15}{M}\right] = \left[\frac{4000\rho - 1178\rho}{321.5\rho}\right] = \frac{2833}{321.5} = 8.8 \text{ cm}$$

Q: Two velocities of the three-particle system are shown in the Fig. 1. If the velocity of the center of mass is zero, find the velocity **v** of the 4.0 kg mass. (Ans: (5i-3j) m/s)



Q: As shown in **Figure**, a ball with a mass of 1.0 kg and a speed of 25 m/s hits a vertical wall at an angle of 45° and rebounds with the same speed with the same angle. Find the change in the linear momentum, in $\text{kg}\cdot\text{m/s}$, of the ball.

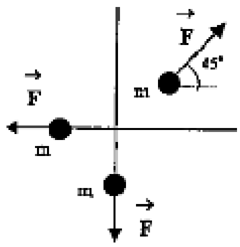


Ans:

$$\Delta P_y = 0$$

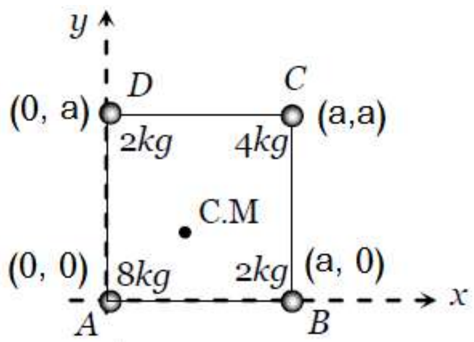
$$\Delta P_x = (-mv - mv)\cos 45^\circ = -2\times 1\times 25\times 0.707 = -35.36$$

Q: What is the magnitude of the acceleration of the center of mass of the system shown in the Figure? Each particle has a mass of 1.00 kg and pulled by a force of 2.0 N in the direction indicated in the Figure. (0.28 m/s^2)



Quizzes

Q: Masses 8, 2, 4, 2 *kg* are placed at the corners *A*, *B*, *C*, *D* respectively of a square *ABCD* of diagonal 80 *cm* . The distance of center of mass from *A* will be:



Answer:
Let corner A of square *ABCD* is at the origin and the mass 8 *kg* is placed at this corner (given in

problem) Diagonal of square $d = \mathrm{a}\sqrt{2} = 80\ \mathrm{cm} \quad \Rightarrow \quad \mathrm{a} = 40\sqrt{2}\ \mathrm{cm}$.

Let $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$ are the position vectors of respective masses

$$\vec{r}_1 = 0\hat{i} + 0\hat{j}, \quad \vec{r}_2 = a\hat{i} + 0\hat{j}, \quad \vec{r}_3 = a\hat{i} + a\hat{j}, \quad \vec{r}_4 = 0\hat{i} + a\hat{j}$$

From the formula of centre of mass

$$\vec{r} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4}{m_1 + m_2 + m_3 + m_4} = 15\sqrt{2}i + 15\sqrt{2}\hat{j}$$

$\therefore$ co-ordinates of centre of mass $= (15\sqrt{2}, 15\sqrt{2})$ and co-ordination of the corner $= (0, 0)$

From the formula of distance between two points (x_1, y_1) and (x_2, y_2)

$$\text{distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(15\sqrt{2} - 0)^2 + (15\sqrt{2} - 0)^2} = \sqrt{900} = 30cm$$

Q: The coordinates of the positions of particles of mass 7, 4 and 10 *gm* are (1,5, □□3), (2,5,7) and
(3, 3, □1) *cm* respectively. The position of the center of mass of the system would be
Answer:

Solution: (c) $m_1 = 7gm$, $m_2 = 4gm$, $m_3 = 10gm$ and $\vec{r_1} = (\hat{i} + 5\hat{j} - 3\hat{k})$, $r_2 = (2i + 5j + 7k)$, $r_3 = (3\hat{i} + 3\hat{j} - \hat{k})$

Position vector of center mass

$$\vec{r} = \frac{7(\hat{i} + 5\hat{j} - 3\hat{k}) + 4(2\hat{i} + 5\hat{j} + 7\hat{k}) + 10(3\hat{i} + 3\hat{j} - \hat{k})}{7 + 4 + 10} = \frac{(45\hat{i} + 85\hat{j} - 3\hat{k})}{21}$$

$\Rightarrow \vec{r} = \frac{15}{7}\hat{i} + \frac{85}{21}\hat{j} - \frac{1}{7}\hat{k}$. So coordinates of centre of mass $\left[\frac{15}{7}, \frac{85}{21}, \frac{-1}{7}\right]$.

Q: A body of mass 5 *kg* is moving with a momentum of 10 *kg-m/s*. A force of 0.2 *N* acts on it in the direction of motion of the body for 10 seconds. The increase in its kinetic energy is

Solution : (d) Change in momentum $= P_2 - P_1 = F \times t \Rightarrow P_2 = P_1 + F \times t = 10 + 0.2 \times 10 = 12 kg\text{-}m/s$

Increase in kinetic energy $E = \frac{1}{2m}[P_2^2 - P_1^2]$

$= \frac{1}{2m}[(12)^2 - (10)^2] = \frac{1}{2 \times 5}[144 - 100] = \frac{44}{10} = 4.4 J.$

***Problem* 24.** Two masses of 1*g* and 9*g* are moving with equal kinetic energies. The ratio of the magnitudes of their respective linear momenta is

(a) 1 : 9 (b) 9 : 1 (c) 1 : 3 (d) 3 : 1

Solution : (c) $P = \sqrt{2mE}$ $\therefore$ $P \propto \sqrt{m}$ if E = constant . So $\frac{P_1}{P_2} = \sqrt{\frac{m_1}{m_2}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

Chapter 9
Center of Mass and Linear Momentum

Important Terms

Impulse

The product of the average force acting on an object and the time during which it acts. Impulse is a vector quantity, and can also be calculated by finding the area under a force versus time curve.

Linear momentum

The product of the mass of an object and its velocity. Momentum is a vector quantity, and thus the total linear momentum of a system of objects is the vector sum of the individual momenta of the objects in the system.

Internal forces

The forces which act between the objects of a system

External forces

The forces which act on the objects of a system from outside the system, that is by an agent which is not a part of the system of objects which are being studied.

Center of mass

The point at which the total mass of a system of masses can be considered to be concentrated.

Equations and Symbols (For the whole chapter)

$x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ $\mathbf{p} = m\mathbf{v}$ $\mathbf{J} = \boldsymbol{F}\Delta t = \Delta\mathbf{p} = m\mathbf{v}_f - m\mathbf{v}_0$	**p** = momentum m = mass **v** = velocity **J** = impulse **F** = force Δt = time interval during which a force acts x_{cm} = position of the center of mass of a system of particles x_1 = position of a mass relative to a chosen origin

Basic Requirements:

1. Understand the concept of Impulse
2. Master the law of conservation of linear momentum

9-4 COLLISION, IMPULSE, and Average Force

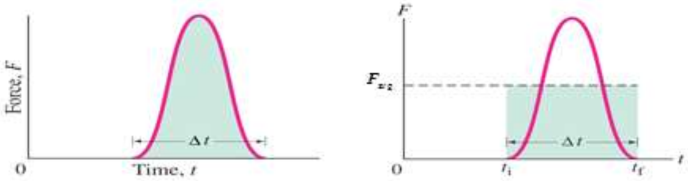
The collision of a ball with a bat collapses part of the ball.

The collision of tennis ball with a racket

When a particle moves freely then interacts with another system for a (brief) period and then moves freely again, it has a definite change in momentum; we define this change as the **impulse** $\boldsymbol{J}$ of the interaction forces:

$$\boldsymbol{J} = \mathbf{P}_f - \mathbf{P}_i = \Delta\mathbf{P}$$

- Impulse is a vector and has the same units as momentum.
- During a collision, objects are deformed due to the large forces involved.
- Since the time of the collision is very short, we need not worry about the exact time dependence of the force, and can use the average force.



The average force gives the same area under the curve.

When we integrate $\boldsymbol{F}_{net} = \dfrac{d\mathbf{P}}{dt}$ we can show:

$$\boldsymbol{J} = \int_{t_i}^{t_f} \boldsymbol{F}\, dt = \Delta\mathbf{P}$$

We can now define the **average force** which acts on a particle during a time interval Δt. It is:

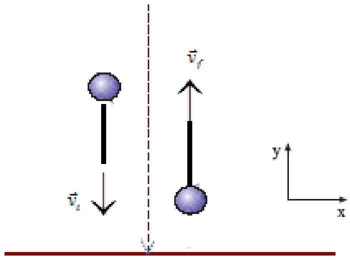
$$\boldsymbol{F}_{\text{avg}} = \frac{\Delta\mathbf{P}}{\Delta t} = \frac{\boldsymbol{J}}{\Delta t}$$

The value of the average force depends on the time interval chosen.

Impulse: is

"*the product of the average force acting on an object and the time during which it acts.*"

Impulse is **a vector quantity**, and can also be calculated by finding the area under a **force versus** time curve.



Example: A 1.0 kg ball falling vertically hits a floor with a velocity of 3.0 m/s and bounces vertically up with a velocity of 2.0 m/s. If the ball is in contact with the floor for 0.1 s, what is the average force on the floor by the ball?

Answer: Define the coordinates as given in the figure, so $\vec{v}_i = -3.0\,\hat{j}$ and $\vec{v}_f = 2.0\,\hat{j}$. The change of linear momentum of the ball is given by:

$$\Delta\mathbf{P} = m\,\Delta\vec{v} = m\left(\vec{v}_f - \vec{v}_i\right) = (1)\left[2.0\,\hat{j} - (-3.0\,\hat{j})\right] = 5.0\,\hat{j}$$

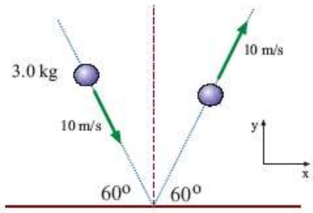
Note: In this type of problem be very careful of the **signs**.

The average force on the floor **by the ball** is given by:

$$\boldsymbol{F}_{\text{avg}} = \frac{\Delta\mathbf{P}}{\Delta t} = \frac{5.0\,\hat{j}}{0.1} = 50\text{ N}\,\hat{j} \qquad \textbf{(Downward)}$$

The average force on the floor **on the ball** is given by:

$$\boldsymbol{F}_{\text{avg}} = \frac{\Delta\mathbf{P}}{\Delta t} = \frac{5.0\,\hat{j}}{0.1} = 50\text{ N}\,\hat{j} \qquad \textbf{(Upward)}$$



Example: A 3.0 kg ball strikes a wall with a speed of 10 m/s at an angle of 60° with the surface. It bounces off with the same speed and angle, as shown in the figure. If the ball is in contact with the wall for 0.20 s, what is the average force exerted on the ball by the wall?

Answer:

The average force is defined as $\boldsymbol{F}_{\text{avg}} = \frac{\Delta\mathbf{P}}{\Delta t}$, so first find the change in momentum of the ball. Since the ball has the same speed before and after bouncing from the wall, it is clear that its *x* velocity (see the coordinate system in the figure) stays the same and so the *x* momentum stays the same. But the *y* momentum does change. The initial *y* velocity is

$$v_{iy} = -(10\,\tfrac{\text{m}}{\text{s}})\sin 60^\circ = -8.7\,\tfrac{\text{m}}{\text{s}}$$

and the final *y* velocity is

$$v_{fy} = +(10\,\tfrac{\text{m}}{\text{s}})\sin 60^\circ = +8.7\,\tfrac{\text{m}}{\text{s}}$$

so the change in *y* momentum is

$$\Delta p_y = mv_{fy} - mv_{iy} = m(v_{fy} - v_{iy}) = (3.0\,\text{kg})(8.7\,\tfrac{\text{m}}{\text{s}} - (-8.7\,\tfrac{\text{m}}{\text{s}})) = 52\,\tfrac{\text{kg}\cdot\text{m}}{\text{s}}$$

The average *y* force **on the ball** is

$$\overline{F}_y = \frac{\Delta p_y}{\Delta t} = \frac{I_y}{\Delta t} = \frac{(52\,\frac{\text{kg}\cdot\text{m}}{\text{s}})}{(0.20\,\text{s})} = 2.6\times10^2\,\text{N}$$

Since $\overline{\boldsymbol{F}}$ has no *x* component, the average force exerted on the ball by the wall has magnitude 2.6×10^2 N and points in the *y*-direction (away from the wall).

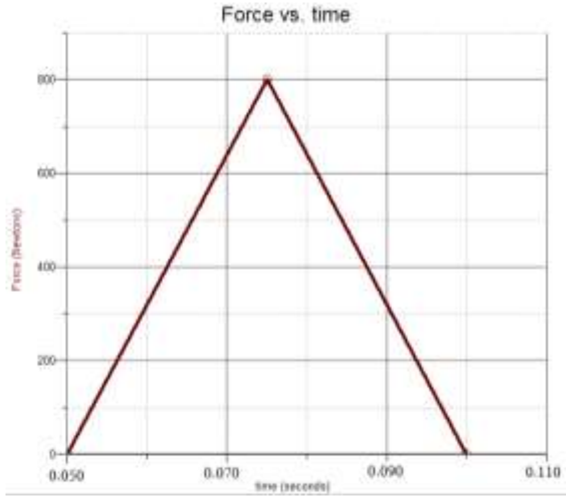
Q: A body of mass 5 *kg* is moving with a momentum of 10 *kg-m/s*. A force of 0.2 *N* acts on it in the direction of motion of the body for 10 seconds. The increase in its kinetic energy is

Solution:

Change in momentum $= P_2 - P_1 = F\times t \Rightarrow P_2 = P_1 + F\times t = 10 + 0.2\times10 = 12 kg\text{-}m/s$

Increase in kinetic energy $E = \frac{1}{2m}[P_2^2 - P_1^2] = \frac{1}{2m}[(12)^2 - (10)^2] = \frac{1}{2\times5}[144-100] = \frac{44}{10} = 4.4J.$

Example: A 2-kg block slides along a floor of negligible friction with a speed of 20 m/s when it collides with a 3-kg block, which is initially at rest. The graph below represents the force exerted on the 3-kg block by the 2-kg block as a function of time.



Find
(a) the initial momentum of the 2-kg block,
(b) the impulse exerted on the 3-kg block, and
(c) the momentum of the 2-kg block immediately after the collision.

Solution:
(a) The initial momentum of the 2-kg block is

$$p = mv = (2kg)\left(20\,\frac{\text{m}}{\text{s}}\right) = 40\,\frac{\text{kg m}}{\text{s}} \text{ to the right.}$$

(b) When the two blocks collide, they exert equal and opposite forces on each other for the same time interval, and thus exert equal and opposite impulses (**F**Δt) on each other. The impulse exerted on the 3-kg block can be found by finding the area under the force vs. time graph:

$$\textit{Impulse} = \textit{area under F vs. t graph} = \tfrac{1}{2}(0.1-0.05)\,(800) = 20 \text{ N.s}.$$

(c) Since the 2-kg block is experiences – 20 N s of impulse, which acts in the opposite direction of the motion of the block, the impulse causes the block to lose momentum. This loss (change) in momentum is equal to the impulse exerted on the block. Thus, by the impulse-momentum theorem,

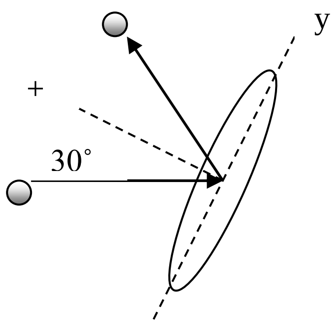
$$\textit{Impulse} = \textit{change in momentum}$$

$$F\Delta t = mv_f - mv_i$$

$$mv_f = mv_i + (F\Delta t) = 40\,\frac{\text{kg m}}{\text{s}} + (-20Ns) = 20\,\frac{\text{kg m}}{\text{s}}$$

Note that a N s is equivalent to $\frac{\text{kg m}}{\text{s}}$.

Example: The photographs shows a tennis ball striking a tennis racquet, applying an impulse to it. Suppose the 0.1 kg ball strikes the racquet with a velocity of 60 m/s at an angle of 30° from a line which is perpendicular to the face of the racquet and rebounds with a speed of 60 m/s at 30° above the perpendicular line, as shown below. The ball is in contact with the strings of the racquet for 12 milliseconds.



Find

a) the magnitude and direction of the average impulse exerted on the ball by the strings of the racquet, and
b) the magnitude of the average acceleration of the ball while it is in contact with the strings.

Solution:

a) The strings of the racquet exert a force and thus an impulse which is perpendicular to the face of the racquet, that is, along the +*x* – axis in the figure above. Therefore the change in momentum of the ball is also only along the +*x* – axis:

$$\mathbf{F}\Delta t = \Delta \mathbf{p}_x = m\Delta \mathbf{v}_x = m(\mathbf{v}_{fx} - \mathbf{v}_{0x})$$

where $v_{0x} = -v_0 \cos 30^\circ$ and $v_{0x} = +v_f \cos 30^\circ$. So the impulse is

$$(0.1kg)[60\cos 30^\circ - (-60\cos 30^\circ)]\frac{\text{m}}{\text{s}} = 10.4\,\text{N s}$$

b) Since the force is applied in the +*x* – direction, the average acceleration is must also be directed along the +*x* – axis, that is, there is no acceleration along the *y*-axis.

$$a_x = \frac{\Delta v}{\Delta t} = \frac{[60.0\cos 30^\circ - (-60.0\cos 30^\circ)]}{0.012\,\text{s}} = 8700\,\frac{\text{m}}{\text{s}^2}$$

9-5 Conservation of Linear Momentum

It is well known that you can never create or destroy momentum; all we can do is transfer momentum from one object to another. Therefore, the total momentum of a system of masses isolated from external forces (forces from outside the system) is constant in time. Similar to Conservation of Energy–always true, no exceptions.

Proof: Consider our definition of linear momentum,

$$\boldsymbol{F}_{net} = \frac{d\boldsymbol{P}}{dt}.$$

Suppose that the sum of all external forces acting on a particle or system of particles is zero. In this case, the momentum of the system cannot be changing in time since $\frac{d\boldsymbol{P}}{dt} = 0$ $\Rightarrow$ $\boldsymbol{P} =$ constant. When the sum of all external forces acting on an isolated system is zero, the total momentum of a system cannot change and momentum is said to be conserved, i.e.,

$$\boldsymbol{P}_i = \boldsymbol{P}_f \qquad \text{(closed, isolated system).}$$

$$\Rightarrow \quad m\boldsymbol{v}_i = m\boldsymbol{v}_f$$

In words,

"If no net external force acts on a system of particles, the total linear momentum $\boldsymbol{P}$ of the system cannot change".

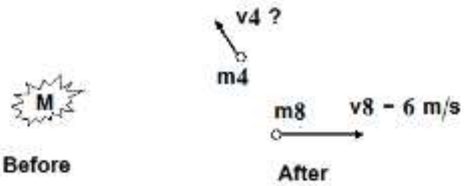
This result is called the **law of conservation of linear momentum** and is an extremely powerful tool in solving problems.

Cautions:

i- Momentum should not be confused with energy. In the sample problems of this note, momentum is conserved but energy is definitely not.

ii- If the component of the net *external* force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.

Example: A bomb of mass M = 12.0 kg, at rest, explodes into two pieces of masses 4.00 kg and 8.00 kg. The velocity of the 8.00 kg mass is $\vec{v}_8$ = 6.00 m/s in the +ve x-direction. Find the change in the kinetic energy.



Answer: We have to calculate the velocity $\vec{v}_4$ using the conservation of the linear momentum i.e.

$$\boldsymbol{P}_i = \boldsymbol{P}_f \quad \Rightarrow \quad M\,\vec{v} = m_4\vec{v}_4 + m_8\vec{v}_8$$

$$0 = 4\times\vec{v}_4 + 8\times 6\hat{i} \quad \Rightarrow \quad \vec{v}_4 = -12\hat{i},$$

- *The negative sign here for $\vec{v}_4$ is essential and makes sense. Initially the system (bomb) is at rest. If the bomb is exploded into two pieces; the two pieces must be moving in the opposite direction to conserve the linear momentum.*

The change in the kinetic energy will be:

$$\Delta K = K_F - K_i = \left[\tfrac{1}{2}m_4\left(\vec{v}_4\right)^2 + \tfrac{1}{2}m_8\left(\vec{v}_8\right)^2\right] - K_{bomb} = \left[\tfrac{4}{2}(12)^2 + \tfrac{8}{2}(6)^2\right] - 0$$

$$= \underline{432\text{ J}}$$

Example: A 6.0 kg body moving with velocity $\vec{v}$ breaks up (explodes) into two equal masses. One mass travels east at 3.0 m/s and the other mass travels north at 2.0 m/s. Calculate the speed $|\vec{v}|$ of the 6.0 kg mass.

Answer: $\vec{v}$ could be calculated using the conservation of the linear momentum i.e.

$$\boldsymbol{P_i} = \boldsymbol{P_f} \quad \Rightarrow \quad M\,\vec{v} = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$6\,\vec{v} = 3\times 3\,\hat{i} + 3\times 2\,\hat{j} \quad \Rightarrow \quad |\vec{v}| = \sqrt{\frac{(3^2+2^2)}{4}} = \sqrt{\frac{13}{4}} = \sqrt{3.25} = 1.8\,\frac{\text{m}}{\text{s}}$$

Example (recoil of a gun)**:** A 4.5 kg gun fires a 0.1 kg bullet with a muzzle velocity of +150 m/s. What is the recoil velocity of the gun?

Answer: There is an external force that acts on this system, gravity. But in the scale of this problem gravity acts only perpendicularly to the velocities of various parts of the system so it has no discernable effect on the momentum along a horizontal axis. (We used the abbreviation *G* for the gun and *b* for the bullet)

$$\boldsymbol{P_i} = \boldsymbol{P_f} \quad \Rightarrow \quad \sum_j m_j \boldsymbol{v}_{\boldsymbol{j,i}} = \sum_k m_k \boldsymbol{v}_{\boldsymbol{k,f}}$$

$$m_G v_{G,i} + m_b v_{b,i} = m_G v_{G,f} + m_b v_{b,f}$$

$$0 + 0 = m_G v_{G,f} + m_b v_{b,f}$$

$$\Rightarrow m_G v_{G,f} = -m_b v_{b,f} \qquad \therefore v_{G,f} = -\frac{(0.1)(150)}{4.5} = -3.3 \text{ m/s.}$$

The negative sign here makes sense. Initially the system (gun + bullet) is at rest. If the bullet is moving forward the gun must be moving in the opposite direction to conserve the linear momentum.

Comments:

1- The gun recoils with an initial velocity of about 3 m/s (≈ 7.5 mph). Depending upon how rapidly the stock of the gun comes to rest against the shooter's shoulder, this could be a heck of a wallop. Unless the gun has a padded stock or the shooter is wearing a padded jacked, this would be like getting hit in the shoulder by a 93 mph fastball (an official major league ball has a mass of 5 oz troy or about 0.16 kg).

2- Also, this is how rockets work! Rocket fuel is thrown out the back of the rocket, causing the rocket to recoil forward. There is NO WAY to make a rocket go forward in space except by throwing mass out the back. Any other means of propulsion would violate Conservation of Momentum.

Incidentally, why is the barrel of a rifle (gun) so long?

Answer: v = a·t $\Rightarrow$ long barrel, more time to accelerate, bigger v

Extra Problems

Example: A child bounces a ball on the sidewalk. The linear impulse delivered by the sidewalk is 2.00 N· s during the 1/800 s of contact. What is the magnitude of the average force exerted on the ball by the sidewalk?

Answer:

The magnitude of the change in momentum of (impulse delivered to) the ball is

$$|\Delta\mathbf{p}| = |\mathbf{I}| = 2.00 \text{ N} \cdot \text{s}.$$

(The direction of the impulse is upward, since the initial momentum of the ball was downward and the final momentum is upward.)

Since the time over which the force was acting was $\Delta t = 1/800$, then from the definition of average force we get:

$$\boldsymbol{F}_{\text{avg}} = \frac{|\boldsymbol{J}|}{\Delta t} = \frac{2.00}{1/800} = 1.60\times 10^3 \text{ N}, \quad \textbf{(Upward)}$$

Example: How good are automobile bumpers? A 1500 kg auto collides elastically with an immovable wall. If the collision lasts 0.150 seconds, the auto was initially traveling +15.0 m/s and rebounds from the wall with a velocity of –2.6 m/s, what is the average force exerted on the bumper?

Answer:

$$\vec{p}_i = mv_i = \left(2.25\times 10^4\, kg\cdot m\cdot s^{-1}\right)$$

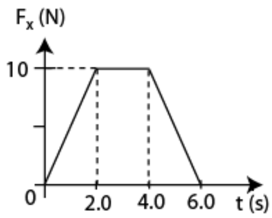
$$\vec{p}_f = mv_f = -\left(0.39\times 10^4\, kg\cdot m\cdot s^{-1}\right)$$

$$\boldsymbol{J} = \Delta\mathbf{P} = \vec{p}_f - \vec{p}_i = -2.64\times 10^4 \text{ kg.m/s}$$

$$\boldsymbol{F}_{\text{avg}} = \frac{\Delta\mathbf{P}}{\Delta t} = \frac{-2.64\times 10^4}{.15} = \underline{-1.76\times 10^5 \text{ N}}$$

• What is the significance of the minus sign here?

Example: A 10.0 kg toy car is moving along the x axis. The only force F_x acting on the car is shown in figure as a function of time (t). At time t = 0 s the car has a speed of 4.0 m/s. What is its speed at time t = 6.0 s?

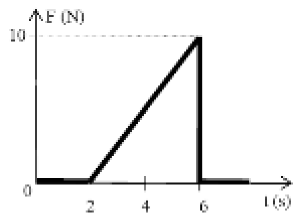


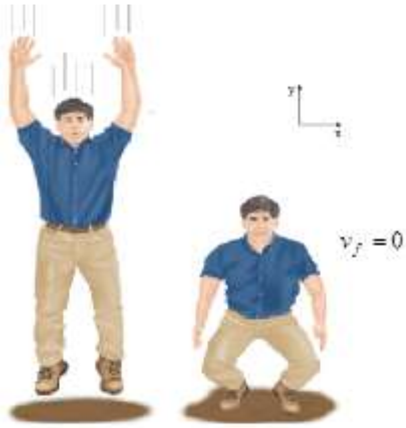
Answer:

$$\boldsymbol{F}_{\text{avg}} = \frac{\Delta\mathbf{P}}{\Delta t} \Rightarrow |\mathbf{P}| = \int \vec{F}dt = \text{ Area under the curve} = \frac{10\times(6.0+2.0)}{2.0} = 40$$

$$\because |\mathbf{P}| = m\left(v_f - v_i\right) \quad \Rightarrow \quad v_f = \frac{|\mathbf{P}|}{m} + v_i = \frac{40}{10} + 4.0 = \underline{8.0 \text{ m/s}}$$

Q7. A 5-kg object is acted upon by a single force in x-direction as shown in the Figure. Find the change of momentum delivered to the object in 6 s. (20 N.s)





Example: Compute the impulse experienced when a 70 kg man lands on firm ground after falling off from a height of 3.0 meters.

Answer: First let's compute the velocity of the man before hitting the ground from a height of 3 meters:

$$v_i = \sqrt{2gh} = \sqrt{2(9.8)(3)} = 7.7\text{m/s}$$

$$\therefore v_i = -7.7\text{m/s}$$

The final velocity:

$$v_f = 0$$

Applying Impulse-Momentum:

$$\boldsymbol{F}_{\text{avg}}\Delta t = \Delta\mathbf{P} = m\left(\mathbf{v}_f - \mathbf{v}_i\right) = 0 - 70(-7.7) = 540 \text{ N.s}$$

Now, if the man lands stiff-legged and stops in a distance of 1.0 cm, find the average force exerted on them by the ground.

First we compute the average velocity over the course of the collision:

$$v_{\text{avg}} = \frac{v_i + v_f}{2} = 3.8 \text{ m.s}^{-1}$$

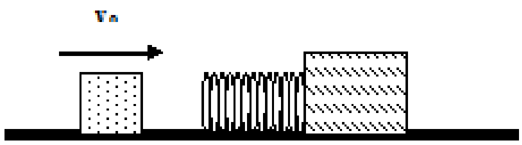
Next the distance over which the collision occurs:

$$v_{\text{avg}} = d\Delta t \Rightarrow \Delta t = \frac{d}{v_{\text{avg}}} = \frac{0.01 \text{ m}}{3.8 \text{ m.s}^{-1}} = 0.0026 \text{ s}$$

So they come to a complete stop in 0.0026 s!

$$\boldsymbol{F}_{\text{avg}}\Delta t = \Delta\mathbf{P} \quad \Rightarrow \quad \boldsymbol{F}_{\text{avg}} = \frac{\Delta\mathbf{P}}{\Delta t} = \frac{540 \text{ N}}{0.0026 \text{ a}} = 2.1\times10^5 \text{ N}$$

Q: A 3.0 kg object, initially at rest explodes into three pieces of equal mass. Two pieces move perpendicular to each other, each with a speed of 10 m/s. What is the speed of the third piece? (14 m/s)



Example: A block of mass *m* is moving on a horizontal frictionless surface with a speed v_o as it approaches a block of mass 2*m* which is at rest and has an ideal spring attached to one side. When the two blocks collide, the spring is completely compressed and the two blocks momentarily move at the same speed, and then separate again, each continuing to move.

(a) Briefly explain why the two blocks have the same speed when the spring is completely compressed.

(b) Determine the speed v_f of the two blocks while the spring is completely compressed.

(c) Determine the kinetic energy of the two blocks as they move together with the same speed.

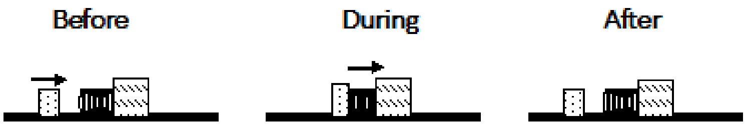
(d) When the spring expands, the blocks are again separated, and the spring returns its compressed potential energy to kinetic energy in the two blocks. On the axes below, sketch a graph of *kinetic energy vs. time* from the time block *m* approaches block 2*m* until the two blocks are separated after the collision.

KE

(e) Write the equations that could be used to solve for the speed of each block after they have separated. It is not necessary to solve these equations for the two speeds.

Answer:

We begin by drawing the situation before, during, and after the collision:



(a) When the spring is completely compressed, the two blocks are at rest relative to each other and must have the same speed. At this point, it is as if they are stuck together immediately after an inelastic collision.

(b) Conservation of momentum:

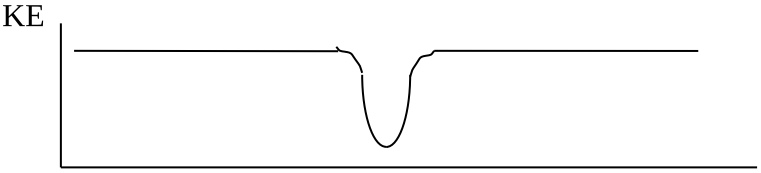
$$mv_0 = (m + 2m)v_f$$

$$v_f = \frac{v_0}{3}$$

(c)

$$K = \frac{1}{2}(3m)\left(\frac{v_0}{3}\right)^2 = \frac{1}{6}mv_0{}^2$$

(d)



The dip in the graph indicates the time during which the spring is compressed. After the blocks separate, all of the kinetic energy is restored to the system.

(e) Since the kinetic energy is the same before the collision and after the blocks have separated, it is as if the blocks have undergone and elastic collision, where both momentum and kinetic energy are conserved. Thus, two equations that could be solved for the speeds after the blocks separate are

$$mv_0 = mv_{f1} + mv_{f2} \text{ and } \frac{1}{2}mv_0{}^2 = \frac{1}{2}mv_{f1}{}^2 + \frac{1}{2}mv_{f2}{}^2$$

Chapter 9
Center of Mass and Linear Momentum

QUICK REFERENCE

Important Terms

impulse
The product of the average force acting on an object and the time during which it acts. Impulse is a vector quantity, and can also be calculated by finding the area under a force versus time curve.

linear momentum
The product of the mass of an object and its velocity. Momentum is a vector quantity, and thus the total linear momentum of a system of objects is the vector sum of the individual momenta of the objects in the system.

internal forces
The forces which act between the objects of a system

external forces
The forces which act on the objects of a system from outside the system, that is by an agent which is not a part of the system of objects which are being studied.

inelastic collision
A collision between two or more objects in which momentum is conserved but kinetic energy is not conserved, such as two railroad cars which collide and lock together.

elastic collision
A collision between two or more objects in which both momentum and kinetic energy are conserved, such as in the collision between two steel balls.

center of mass
The point at which the total mass of a system of masses can be considered to be concentrated.

Equations and Symbols (For the whole chapter)

$x_{cm} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$ $\mathbf{p} = m\mathbf{v}$ $\Delta\mathbf{p} = m(\mathbf{v}_f - \mathbf{v}_0)$	$\mathbf{p}$ = momentum m = mass $\mathbf{v}$ = velocity x_{cm} = position of the center of mass of a system of particles x_1 = position of a mass relative to a chosen origin

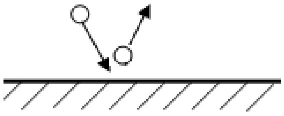
Basic Requirements:

1. Master and understand the classification of collision (elastic, inelastic, perfectly elastic, perfectly inelastic)

9-7 Collisions in one Dimension

In physics, a collision is an isolated process in which two or more objects interact in such a manner as to exert a relatively strong force one each other for a relatively short time. There are two primary types of collisions we study in physics, perfectly elastic and perfectly inelastic. These are two end points along a continuum of possibilities. Because of the mathematical complexity involved in studying collisions that are not perfectly elastic or inelastic we will confine ourselves to the two idealized cases.

<u>Types of collisions</u>



elastic collision : total kinetic energy (KE) is conserved
(KE before = KE after)

superball on concrete: KE just before collision = KE just after (almost!) The Initial KE just before collision is converted to elastic PE as the ball compresses during the first half of its collision with the floor. But then the elastic PE is converted back into KE as the ball un-compresses during the second half of its collision with the floor.

inelastic collision : some KE is lost to thermal energy, sound, etc

perfectly inelastic collision (or totally inelastic collision) : 2 objects collide and stick together

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All collisions between macroscopic (large) objects are inelastic – you always dissipate some KE in a collision. However, you can have an elastic collision between atoms: air molecules are always colliding with each other, but do not lose their KE.

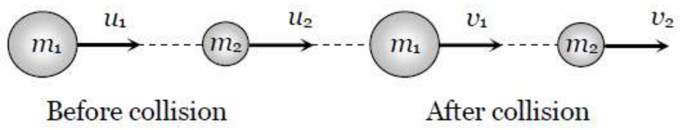
Summary

Collision	**momentum**	**Kinetic energy**
Perfectly Elastic Head on Collision	conserved	conserved
inelastic	conserved	Not conserved
Perfectly inelastic	conserved	Not conserved

1D Collisions

Perfectly Elastic Head on Collision:

Elastic collisions occur only between extremely hard objects. In elastic collisions objects exchange momentum by “bouncing” from each other during the collision process (like billiard balls). It is not necessary for any of the interacting particles to actually touch each other in an elastic collision (as when a proton is scattered, elastically from a nucleus). ***<u>Both momentum and kinetic energy are conserved in elastic collisions</u>***. Elastic collisions are of great interest in physics. Accelerating particles to high speeds and colliding them, elastically, with various targets is an often used method of interrogating atomic scale matter. Let two bodies of masses m_1 and m_2 moving with initial velocities u_1 and u_2 in the same direction and they collide such that after collision their final velocities are v_1 and v_2 respectively (see the figure), then:



$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2, \qquad \boldsymbol{P}_i = \boldsymbol{P}_f$$

$$\tfrac{1}{2}m_1u_1^2 + \tfrac{1}{2}m_2u_2^2 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2, \qquad KE_i = KE_i$$

Solve for v_1 and v_2 we get

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

and

$$v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2 + \left(\frac{2m_1}{m_1 + m_2}\right)u_1$$

In 1D, we represent direction of vectors **p** and **v** with a sign. (+) = right (–) = left

$\longrightarrow$ (+) $v_A = +\,2$ m/s $\Rightarrow$ moving right
$v_B = -\,3$ m/s $\Rightarrow$ moving left

Notation Danger!! Sometimes $v = |\vec{v}| = \text{speed}$ (always positive). But in 1D collision problems, symbol "v" represents *velocity* : v can (+) or (–).

Special cases of head on elastic collision

(i) **If projectile and target are of same mass *i.e.*** $m_1 = m_2$

For the equations:

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2\,, \quad \text{and} \quad v_2 = \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2 + \left(\frac{2m_1}{m_1 + m_2}\right)u_1$$

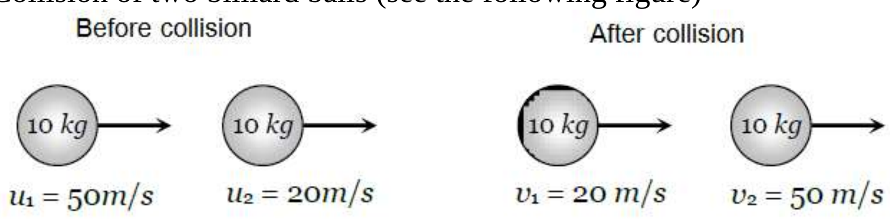
Substituting $m_1 = m_2$ we get

$$v_1 = u_2 \quad \text{and} \quad v_2 = u_1$$

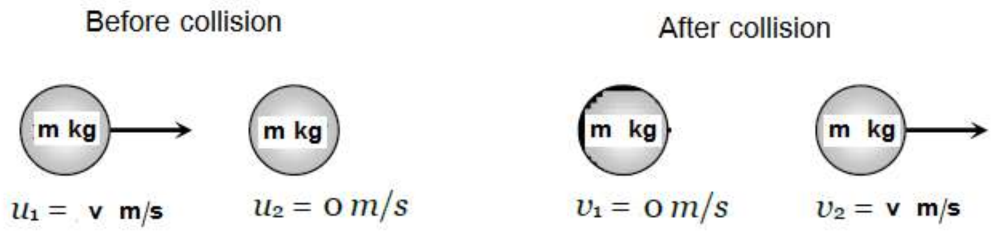
- It means when two bodies of equal masses undergo head on elastic collision, their velocities get interchanged.
- Sub case : $u_2 = 0$ *i.e.* target is at rest

$$v_1 = 0 \text{ and } v_2 = u_1.$$

Example: Collision of two billiard balls (see the following figure)



Example: Collision of two billiard balls, $u_2 = 0$ (see the following figure)



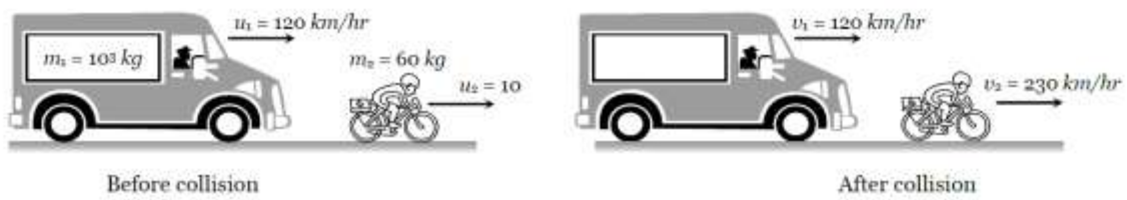
(ii) **If massive projectile collides with a light target *i.e.*** $m_1 >> m_2$

Since $v_1 = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\dfrac{2m_2}{m_1 + m_2}\right)u_2$, and $v_2 = \left(\dfrac{m_2 - m_1}{m_1 + m_2}\right)u_2 + \left(\dfrac{2m_1}{m_1 + m_2}\right)u_1$

Substituting $m_2 = 0$, we get

$$v_1 = u_1 \quad \text{and} \quad v_2 = 2u_1 - u_2$$

Example: Collision of a truck with a cyclist (see the following figure)



- **Sub-case:** $u_2 = 0$ i.*e.* target is at rest $v_1 = u_1$ and $v_2 = 2u_1$.

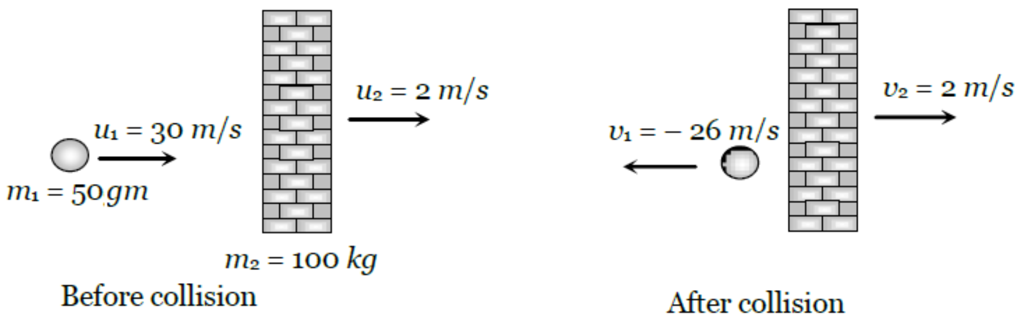
(ii) **If light projectile collides with a very heavy target *i.e.*** $m_1 << m_2$

Since $v_1 = \left(\dfrac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\dfrac{2m_2}{m_1 + m_2}\right)u_2$, and $v_2 = \left(\dfrac{m_2 - m_1}{m_1 + m_2}\right)u_2 + \left(\dfrac{2m_1}{m_1 + m_2}\right)u_1$

Substituting $m_1 = 0$, we get

$$v_1 = -u_1 + 2u_2 \quad \text{and} \quad v_2 = u_2$$

Example: Collision of a ball with a massive wall (see the following figure).



Sub case: $u_2 = 0$ i.e. target is at rest $v_1 = -u_1$ and $v_2 = 0$ *i.e.* the ball rebounds with same speed in opposite direction when it collides with stationary and very massive wall.

Example: A 10 kg ball (m_1) with a velocity of +10 m/s collides head on in an elastic manner with a 5 kg ball (m_2) at rest. What are the velocities after the collision?

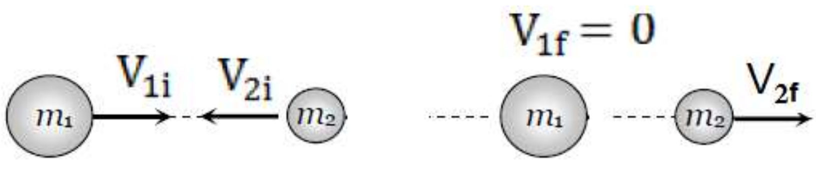
Answer: Here $u_2 = 0$, then

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 = \left(\frac{10-5}{10+5}\right)(+10) = +3.33 \text{ m/s}$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 = \left(\frac{2\times 10}{10+5}\right)(+10) = 13.33 \text{ m/s},$$

Which ball experiences the greater change in momentum?

Example: Two metallic solid spheres **approach each other head-on** with the same speed of 3.00 m/s and collide elastically. After the collision, one of the spheres, whose mass is 600 g, remains at rest. What is the speed of the two-sphere center of mass?



Before collision After collision

Answer: Note that the magnitude $V_{1i} = V_{2i}$ and $V_{1f} = 0$.

$$V_{1f} = \frac{m_1 - m_2}{m_1 + m_2}V_{1i} + \frac{2m_2}{m_1 + m_2}(-V_{2i}) = 0$$

$$\Rightarrow m_1 - m_2 - 2m_2 = 0 \ \Rightarrow m_2 = \frac{m_1}{3} = 200\text{ g}$$

$$m_1|V_{1i}| - m_2|V_{2i}| = V_{com}(m_1 + m_2) \ \Rightarrow \ V_{com} = 1.5 \text{ m/s}$$

Example: A 20.0 g bullet moving vertically upward at 1.00×10^3 m/s strikes and passes through the center of mass of a 10.0 kg block initially at rest, as shown in **Figure**. To what maximum height does the block rise after the bullet emerges from the block with a speed of 4.00×10^2 m/s vertically upward. Ignore air resistance.

Before **After**

Answer: Define b = Bullet, B = block,

Initial momentum: $\vec{P}_i = m\vec{v}_{bi} + M\vec{V}_{Bi} = m\vec{v}_{bi}$ (i)

Final momentum: $\vec{P}_f = m\vec{v}_{bf} + M\vec{V}_{Bf}$ (ii)

Conservation of momentum $\vec{P}_i = \vec{P}_f \Rightarrow$:

$$V_{Bf} = \frac{m(v_{bi} - v_{bf})}{M} = \frac{0.02 \times (10^3 - 4 \times 10^2)}{10} = 1.2 \text{ m/s}$$

Conservation of mechanical energy implies:

$$\cancel{M}gh = \frac{1}{2}\cancel{M}V_{Bf}^2$$

$$h = \frac{(V_{Bf})^2}{2g} = \frac{(1.2)^2}{2 \times 9.8} = 0.0735 = 7.35 \text{ cm}$$

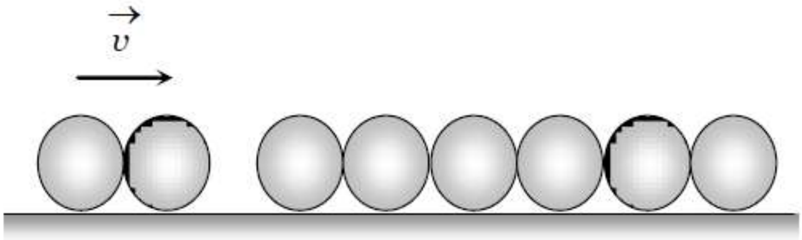
Inelastic collisions:

Perfectly inelastic collisions occur when the objects involved in the collision stick together. ***<u>Momentum is conserved in inelastic collisions but kinetic energy is not</u>***. For a two-body system:

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2)v_f, \qquad \boldsymbol{P}_i = \boldsymbol{P}_f$$

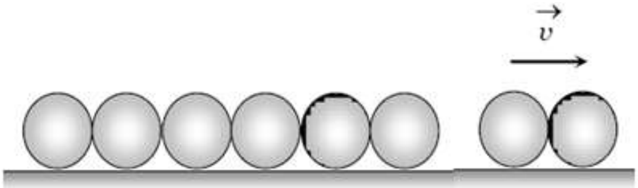
In either case we are generally interested in studying the state of the system before, during, and after the collision process.

Q: Six identical balls are lined in a straight groove made on a horizontal frictionless surface as shown. Two similar balls each moving with a velocity *v* collide with the row of 6 balls from left. What will happen?

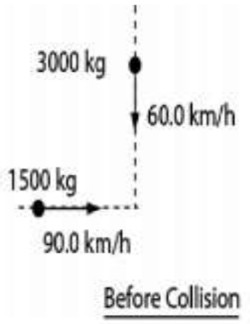


(a) One ball from the right rolls out with a speed 2*v* and the remaining balls will remain at rest.
(b) Two balls from the right roll out with speed *v* each and the remaining balls will remain stationary.
(c) All the six balls in the row will roll out with speed *v*/6 each and the two colliding balls will come to rest.
(d) The colliding balls will come to rest and no ball rolls out from right.

Solution: (b) Only this condition satisfies the law of conservation of linear momentum and we will have:



Example: A 1500 kg car traveling at 90.0 km/h east collides with a 3000 kg car traveling at 60.0 km/h south. The two cars stick together after the collision (see Figure). What is the speed of the cars after collision?



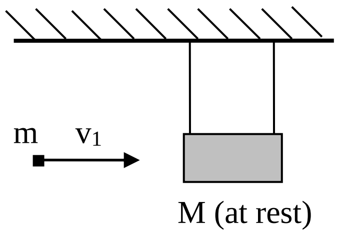
Answer: Conservation of momentum $\vec{P}_i = \vec{P}_f$ in case of perfectly inelastic collision implies**:**

$$m_1\vec{v}_1 + m_2\vec{v}_2 = M\ \vec{v}$$

$$1500\times 90\ \hat{\text{i}} - 3000\times 60\ \hat{\text{j}} = (1500+3000)\ \vec{v}$$

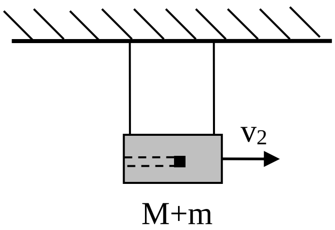
$$\Rightarrow |\vec{v}| = \sqrt{\left(30^2+40^2\right)} = 50\ \frac{\text{km}}{\text{hour}} = 13.9\ \underline{\text{m/s}}$$

Example of Conservation of Energy and Momentum: The Ballistic Pendulum. The ballistic pendulum is a simple device which can accurately measure the speed of a bullet. It consists of a block of wood hanging from some strings. When a bullet is fired into the block, the kick from the bullet cause the block to swing upward. From the height of the swing, the speed of the bullet can be determined.



bullet of mass m, with unknown initial velocity v_1 , is fired into a large wooden block of mass M, hanging at rest from strings.

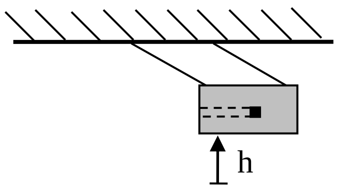
p_{ot} = m v1 (0)



Immediately after collision, bullet is buried in block, but block has not yet had time to move. The impulse from bullet gives block+bullet a velocity v_2.

Momentum conservation $\Rightarrow$ $m\cdot v_1 = (M + m)\cdot v_2$ (1)

Momentum is conserved, but KE is not. Most of the bullet's initial KE has been converted to thermal energy: bullet and block get hot. Some KE is left over: $\text{KE} = \frac{1}{2}(\text{m}+\text{M})\,\text{v}_2^{\,2}$



Block+bullet rise to max height h, which is measured.

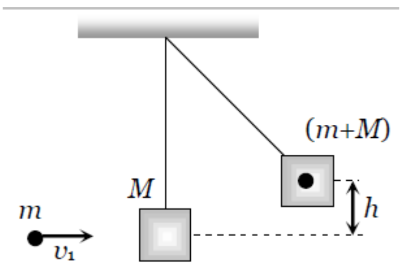
Conservation of energy $\Rightarrow$

$$\text{KE}_\text{i} + \underset{0}{\text{PE}_\text{i}} = \underset{0}{\text{KE}_\text{f}} + \text{PE}_\text{f}$$

$$\tfrac{1}{2}\,(\text{M}+\text{m})\,\text{v}_2^{\,2} = (\text{M}+\text{m})\,\text{g}\,\text{h} \qquad (2)$$

Now have 2 equations [(1) and (2)] in two unknowns (v_1 and v_2). So you can solve for the velocity of the bullet v_1 terms of the knowns (m, M, g, and h).

Example: A bullet of mass m moving with a velocity v_1 strikes a suspended wooden block of mass M as shown in the figure and sticks to it. If the block rises to a height h, calculate the initial velocity of the bullet.



Answer:

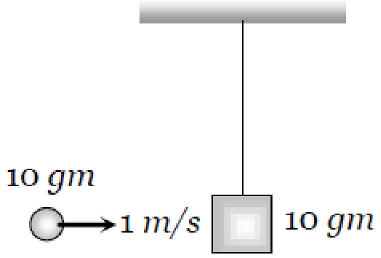
By the conservation of momentum $mv_1 = (m+M)V$ (i)

and if the system goes up to height h then $V = \sqrt{2gh}$ (ii)

From (ii) to (i) one gets:

$$mv_1 = (m+M)\sqrt{2gh} \quad \Rightarrow \quad v_1 = \left(\frac{m+M}{m}\right)\sqrt{2gh}$$

Example: A mass of $10gm$, moving horizontally with a velocity of $100cm/\sec$, strikes the bob of a pendulum and strikes to it. The mass of the bob is also $10gm$ (see fig.) The maximum height to which the system can be raised is ($g = 10\ m/\sec^2$)



Answer:

By the conservation of momentum,

Momentum of the bullet = Momentum of system

$$\Rightarrow 10 \times 1 = (10+10) \times v \Rightarrow v = \frac{1}{2} m/s$$

Now maximum height reached by system $H_{\max} = \dfrac{v^2}{2g} = \dfrac{(1/2)^2}{2 \times 10} m = 1.25$ cm .

Example: A wooden block of mass M is suspended by a cord and is at rest. A bullet of mass m, moving with a velocity v pierces through the block and comes out with a velocity $v/2$ in the same direction. If there is no loss in kinetic energy, then upto what height the block will rise

Answer:

By the conservation of momentum

Initial momentum = Final momentum

$$mv + M \times 0 = m\frac{v}{2} + M \times V \Rightarrow V = \frac{m}{2M}v$$

If block rises upto height h then $h = \dfrac{V^2}{2g} = \dfrac{(mv/2M)^2}{2g} = \dfrac{m^2v^2}{8M^2g}$.

Example: A 10.0 g bullet is stopped in a block of wood (m = 5.00 kg). The speed of the bullet–plus–wood combination immediately after the collision is 0.600 m/s. What was the original speed of the bullet?

Answer:

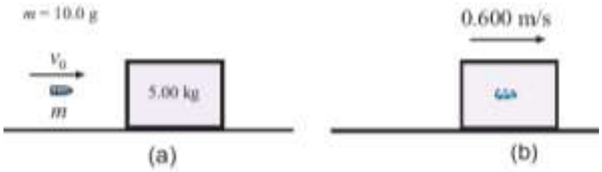


Figure 7.3: Collision in Example 5. (a) Just before the collision. (b) Just after.

A picture of the collision just before and after the bullet (quickly) embeds itself in the wood is given in Fig. 7.3. The bullet has some initial speed v_0 (we don't know what it is.)

The collision (and embedding of the bullet) takes place very rapidly; for that brief time the bullet and block essentially form an isolated system because any external forces (say, from friction from the surface) will be of no importance compared to the enormous forces between the bullet and the block. So the total momentum of the system will be conserved; it is the same before and after the collision.

In this problem there is only motion along the x axis, so we only need the condition that the total x momentum (P_x) is conserved.

Just before the collision, only the bullet (with mass m) is in motion and its x velocity is v_0. So the initial momentum is

$$P_{i,x} = mv_0 = (10.0 \times 10^{-3}\,\mathrm{kg})v_0$$

Just after the collision, the bullet–block combination, with its mass of $M + m$ has an x velocity of 0.600 m/s. So the final momentum is

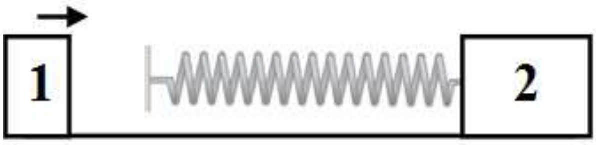
$$P_{f,x} = (M+m)v = (5.00\,\mathrm{kg} + 10.0 \times 10^{-3}\,\mathrm{kg})(0.600\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 3.01\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}}$$

Since $P_{i,x} = P_{f,x}$, we get:

$$(10.0 \times 10^{-3}\,\mathrm{kg})v_0 = 3.01\,\tfrac{\mathrm{kg\cdot m}}{\mathrm{s}} \qquad \Longrightarrow \qquad v_0 = 301\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The initial speed of the bullet was 301 m/s.

Q: In the **Figure**, block 1 (mass 2.0 kg) is moving rightward at 9.0 m/s and block 2 (mass 4.0 kg) is at rest. The surface is frictionless, and a spring with a spring constant of 1120 N/m is fixed to block 2. When the blocks collide, the compression of the spring is maximum at the instant the blocks have the same velocity. Find the maximum compression, $x_{\max}$ in meters.



Ans: The linear momentum of the two blocks and the spring system is conserved, and the mass of the spring is negligible. Choose rightward as positive direction and suppose when the compression of the spring is maximum, the velocity of the blocks is v, we have:

$$(2.0\mathrm{kg})(9\ \mathrm{m/s}) + (4.0\mathrm{kg})(0.0\mathrm{m/s}) = (2.0\mathrm{kg} + 4.0\mathrm{kg})v$$

Solving for v, yields $\boxed{v = 3\,m/s}$

Because of the compression of the spring, the total kinetic energy of the system decreased, The conservation of mechanical energy requires:

$$\Delta E = \Delta K.E. + \Delta P.E. = K_f - K_i + (PI_f - PI_i) = 0$$

H.W. Calculate K_f, K_i, PI_f and PI_i.

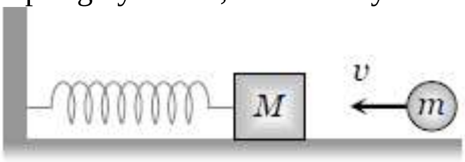
which gives:

$$\frac{1}{2}kx_{\max}{}^{2} = \frac{1}{2}(2.0\text{kg})(9\text{m/s})^2 + \frac{1}{2}(4.0\text{kg})(0\text{m/s})^2 - \left[\frac{1}{2}(2.0\text{kg} + 4.0\text{kg})(3\text{m/s})^2\right]$$

Substituting k = 1120 N/m into the above equation yields the maximum compression of the spring:

$$\boxed{x_{\max} = 0.31\,\text{m}}$$

Q: A body of mass 2*kg* is placed on a horizontal frictionless surface. It is connected to one end of a spring whose force constant is 250 N/m. The other end of the spring is joined with the wall. A particle of mass 0.15 kg moving horizontally with speed *v* sticks to the body after collision. If it compresses the spring by 10 cm, the velocity of the particle is



Answer: By the conservation of momentum

Initial momentum of particle = Final momentum of system: $m\,v = (m + M)\,V$

The velocity of system

$$V = \frac{mv}{(m + M)}$$

Now the spring compresses due to kinetic energy of the system so by the conservation of energy

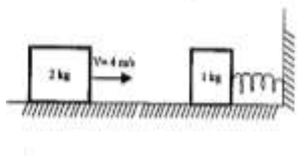
$$\frac{1}{2}kx^2 = \frac{1}{2}(m + M)V^2 = \frac{1}{2}(m + M)\left(\frac{mv}{m + M}\right)^2$$

$$\Rightarrow kx^2 = \frac{m^2v^2}{m + M} \Rightarrow v = \sqrt{\frac{kx^2(m + M)}{m^2}} = \frac{x}{m}\sqrt{k(m + M)}$$

Putting m = 0.15 *kg*, M = 2 kg, k = 250 N/m, x = 0.1 *m* we get v = 15 m/s.

Extra problems

Q: A 1.0-kg block at rest on a horizontal frictionless surface is connected to a spring (k = 200 N/m) whose other end is fixed (see figure). A 2.0-kg block moving at 4.0 m/s collides with the 1.0-kg block. If the two blocks stick together after the one-dimensional collision, what maximum compression of the spring does occur when the blocks momentarily stop?



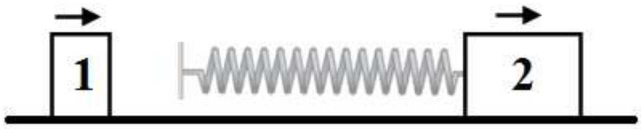
Answer:

Consrevation of momentum $\Rightarrow \quad mv = (m+M)V \quad \Rightarrow \quad V = \dfrac{2\times 4}{(2+1)} \approx 2.67\,\dfrac{\text{m}}{\text{s}}$

Conservation of K.E. after collision $\Rightarrow \quad \dfrac{1}{2}(m+M)V^2 = \dfrac{1}{2}k\,x^2$

$$\Rightarrow \quad x = V\sqrt{\frac{(m+M)}{k}} = 2.67\sqrt{\frac{3}{200}} = \underline{0.33\text{ m}}$$

Q: In the following figure, block 1 (mass 2.0 kg) is moving rightward at 10 m/s and block 2 (mass 5.0 kg) is moving rightward at 3.0 m/s. The surface is frictionless, and a spring with a spring constant of 1120 N/m is fixed to block 2. When the blocks collide, the compression of the spring is maximum at the instant the blocks have the same velocity. Find the maximum compression.



Solution: The linear momentum of the two blocks and the spring system is conserved, and the mass of the spring is negligible. Choose rightward as positive direction and suppose when the compression of the spring is maximum, the velocity of the blocks is v, we have

$$(2.0kg)(10m/s)+(5.0kg)(3.0m/s)=(2.0kg+5.0kg)v$$

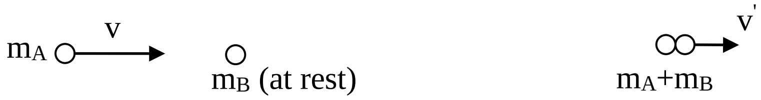
Solving for v, yields $\quad v = 5m/s$

Because of the compression of the spring, the total kinetic energy of the system decreased, gives us

$$\frac{1}{2}kx_{\max}{}^2 = \frac{1}{2}(2.0kg)(10m/s)^2 + \frac{1}{2}(5.0kg)(3m/s)^2 - \left[\frac{1}{2}(2.0kg+5.0kg)(5m/s)^2\right]$$ Substituting k=1120N/m into the above equation yields the maximum compression of the spring:

$$x_{\max} = 0.25m \qquad \text{(Answer)}$$

1D collision example: 2 objects, A and B, collide and stick together (a perfectly inelastic collision). Object A has initial velocity v, object B is initially at rest. What is the final velocity v' of the stuck-together masses?



Before **After**

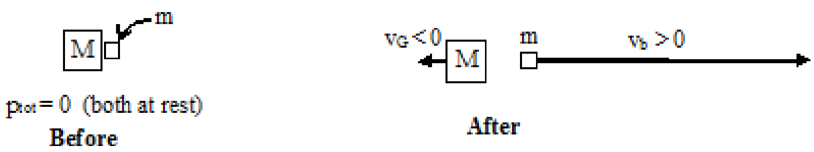
$$p_{tot,before} = p_{tot,after}$$

$$m_A \underbrace{v_A}_{v} + m_B \underbrace{v_B}_{0} = (m_A + m_B)\, v'$$

$$m_A v = (m_A + m_B)\, v' \Rightarrow v' = \left(\frac{m_A}{m_A + m_B}\right) v \Rightarrow$$

Notice that $v' < v$, since $m_A/(m_A+m_B) < 1$.

Another 1D collision example (recoil of a gun). A gun of mass *M* fires a bullet of mass *m* with velocity v_b . What is the recoil velocity v_G of the gun?



$$p_{tot(before)} = 0 = p_{tot(after)}$$

$$0 = M\, v_G + m\, v_b$$

$$M\, v_G = -m\, v_b$$

$$v_G = -\left(\frac{m}{M}\right) v_b$$

$v_b = 500$ m/s, $m = 10$ gram $= 0.01$ kg, $M = 3$ kg $\Rightarrow v_G = -\dfrac{0.010}{3} \cdot 500 = -1.7$ m/s

Quite a kick! This is how rockets work! Rocket fuel is thrown out the back of the rocket, causing the rocket to recoil forward. There is NO WAY to make a rocket go forward in space except by throwing mass out the back. Any other means of propulsion would violate Conservation of Momentum.

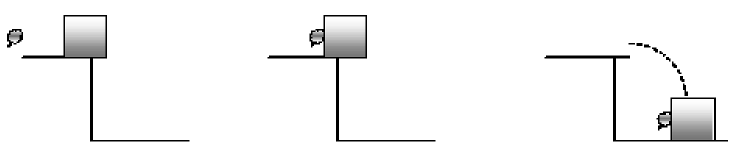
Example 4: A lump of clay (m_1 = 0.2 kg) moving horizontally with a speed v_{o1} = 16 m/s strikes and sticks to a wood block (m_2 = 3 kg) which is initially at rest on the edge of a horizontal table of height h = 1.5 m. Neglecting friction, find

(a) the horizontal distance x from the edge of the table at which the clay and block strike the floor, and

(b) the total momentum of the clay and block just before they strike the floor.

Solution: The sketches below represent the system before, during, and after the collision.

Before *During* *After*



(a) Before we can find the horizontal distance the clay and block travel we need to find their speed v_f as they leave the edge of the table. Momentum is conserved in this inelastic collision:

$$\mathbf{p}_o = \mathbf{p}_f$$
$$m_1\mathbf{v}_{o1} = (m_1 + m_2)\mathbf{v}_f$$
$$v_f = \frac{m_1 v_{o1}}{(m_1 + m_2)} = \frac{(0.2\,\text{kg})(16\,\frac{\text{m}}{\text{s}})}{(0.2\,\text{kg} + 3.0\,\text{kg})} = 1\frac{\text{m}}{\text{s}}$$

Now the clay and block have become a projectile which is launched horizontally. We can find the time of flight by using the height:

$$h = \frac{1}{2}gt^2$$
$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(1.5\,\text{m})}{10\,\frac{\text{m}}{\text{s}^2}}} = 0.55\text{s}$$

Then the horizontal distance traveled is $x = v_f t = (1\frac{m}{s})(0.55\,s) = 0.55\,m$

(b) The momentum of the clay and block just before it strikes the ground can be found by finding the horizontal and vertical components of the momentum:

$$p_x = mv_x = (3.2)(1) = 3.2 \quad \text{kg.m/s}$$
$$p_y = mv_y = m(gt) = (3.2)(10)(0.55) = 17.6 \quad \text{kg.m/s}$$

By the Pythagorean theorem, the magnitude of the momentum can be found by

$$p = \sqrt{p_x{}^2 + p_y{}^2} = \sqrt{(3.2)^2 + (17.6)^2} = 17.9\,\text{kg.m/s}$$

The angle at which the momentum is directed is

$$\theta = \tan^{-1}\left[\frac{p_y}{p_x}\right] = \tan^{-1}\left[\frac{17.6}{3.2}\right] = 80° \quad \text{below the horizontal.}$$

Problem . A particle of mass $1g$ having velocity $3\hat{i} - 2\hat{j}$ has a glued impact with another particle of mass $2g$ and velocity as $4\hat{j} - 6\hat{k}$. Velocity of the formed particle is

(a) $5.6 ms^{-1}$ (b) 0 (c) $6.4 ms^{-1}$ (d) $4.6 ms^{-1}$

Solution : (d) By conservation of momentum $m\vec{u}_1 + m_2\vec{u}_2 = (m_1 + m_2)\vec{V}$

$$\therefore \ \vec{V} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2} = \frac{1(3\hat{i} - 2\hat{j}) + 2(4\hat{j} - 6\hat{k})}{m_1 + m_2} = \frac{3\hat{i} + 6\hat{j} - 12\hat{k}}{(1+2)} = \hat{i} + 2\hat{j} - 4\hat{k}$$

$$|\vec{V}| = \sqrt{(1)^2 + (2)^2 + (-4)^2} = \sqrt{1 + 4 + 16} = 4.6 ms^{-1}.$$

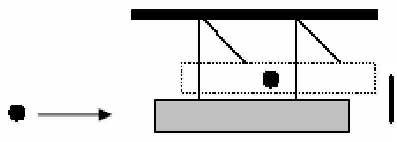
Problem 109. A neutron having mass of $1.67\times10^{-27}kg$ and moving at $10^8 m/s$ collides with a deutron at rest and sticks to it. If the mass of the deutron is $3.34\times10^{-27}kg$; the speed of the combination is

(a) $2.56\times10^3 m/s$ (b) $2.98\times10^5 m/s$ (c) $3.33\times10^7 m/s$ (d) $5.01\times10^9 m/s$

Solution : (c) $m_1 = 1.67\times10^{-27}kg$, $u_1 = 10^8 m/s$, $m_2 = 3.34\times10^{-27}kg$ and $u_2 = 0$

Speed of the combination $V = \frac{m_1u_1 + m_2u_2}{m_1 + m_2} = \frac{1.67\times10^{-27}\times10^8 + 0}{1.67\times10^{-27} + 3.34\times10^{-27}} = 3.33\times10^7 m/s.$

Q: A 0.0025 kg bullet, traveling at a velocity of +425 m/s imbeds itself in the wooden block of a ballistic pendulum. If the wooden block has a mass of 0.200 kg, to what height does the bullet-block combination rise?



Answer:
Since the bullet and the block "stick together" after the collision, this is an example of a **perfectly inelastic** collision. Momentum is conserved but kinetic energy is not.
Conserve momentum:

$$mv_{bullet_i} + Mv_{block_i} = (m+M)v_f$$

$$mv_{bullet_i} = (m+M)v_f$$

$$v_f = \frac{mv_{bullet\,i}}{m+M} = 5.25 m\cdot s^{-1} = v_{aftercollision}$$

After the collision energy is conserved as long as non-conservative forces do not act on the system, hence:

$$KE_{aftercollision} = PE_f$$

$$\frac{1}{2}(m+M)v_{ac}^{\ 2} = (m+M)gh$$

$$h = \frac{v_{ac}^{\ 2}}{2g} = 1.4m$$

What is the percentage of mechanical energy lost in this problem? Calculate $\left|\frac{K_f - K_i}{K_i}\right| \times 100$

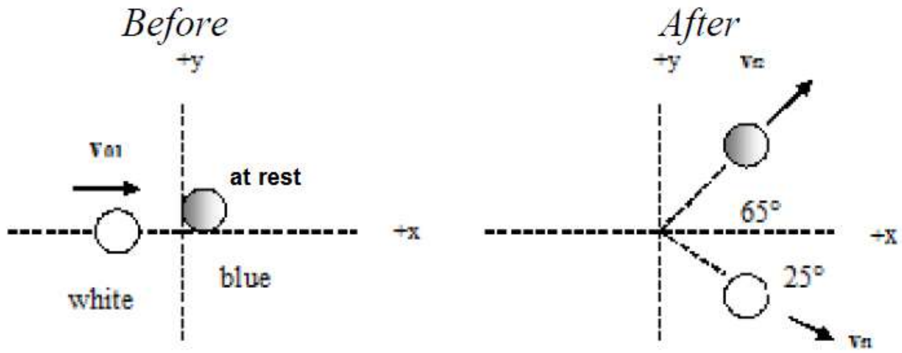
Chapter 9
Center of Mass and Linear Momentum

Very Important Statement:

"In Two–Dimensional collisions, we have to apply the conservation of momentum in both directions, i.e x- and y- directions."

9-8 Collisions in Two–Dimensional

Example: The diagram below shows a collision between a white pool ball (m_1 = 0.3 kg) moving at a speed v_{01} = 5 m/s in the +*x* direction and a blue pool ball (m_2 = 0.6 kg) which is initially at rest. The collision is not head-on, so the balls bounce off of each other at the angles shown. Find the final speed of each ball after the collision.



Solution: The components of the momenta before and after the collision are conserved. Writing the *x* – components of the momentum before and after the collision:

$$\mathbf{p}_{01x} + \mathbf{p}_{02x} = \mathbf{p}_{f1x} + \mathbf{p}_{f2x}$$

$$m_1\mathbf{v}_{01x} + m_2\mathbf{v}_{02x} = m_1\mathbf{v}_{f1x} + m_2\mathbf{v}_{f2x}$$

$$(0.3kg)(5\frac{\text{m}}{\text{s}}) = (0.3kg)(v_{f1}\cos 25 + (0.6\,\text{kg})(v_{f2}\cos 65)$$

Here we have two unknowns, v_{f1} and v_{f2}, and only one equation so far. When we write the conservation of momentum equation for the y-components of the momentum of each ball, we see that the total momentum in the y direction is zero before the collision, and thus must be zero after the collision.

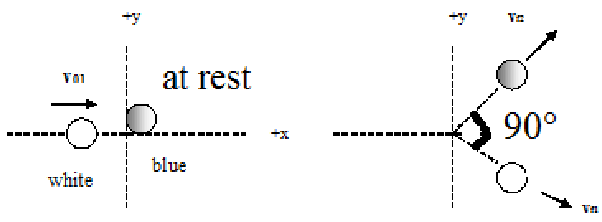
$$\mathbf{p}_{01y} + \mathbf{p}_{02y} = \mathbf{p}_{f1y} + \mathbf{p}_{f2y}$$

$$0 = m_1\mathbf{v}_{f1y} + m_2\mathbf{v}_{f2y}$$

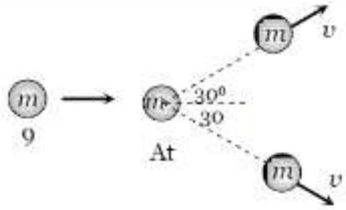
$$0 = -(0.3kg)(v_{f1}\sin 25) + (0.6kg)(v_{f2}\sin 65)$$

Solving these two equations simultaneously for the unknown speeds gives v_{f1} = 1.8 m/s and v_{f2} = 4 m/s.

Note: In any two-dimensional elastic collision in which one mass is at rest, the angle between the two objects after the collision will be 90°.

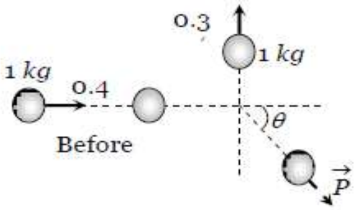


Example: A ball moving with velocity of 9 m/s collides with another similar stationary ball. After the collision both the balls move in directions making an angle of 30° with the initial direction. Calculate their speed after the collision.



Answer: Initial horizontal momentum of the system = $m \times 9$
Final horizontal momentum of the system = $2mv \cos 30^o$
According to law of conservation of momentum, $m \times 9 = 2mv \cos 30^o$
$\Rightarrow v = 5.2$ m/s

Example: A ball of mass 1 kg , moving with a velocity of 0.4 m/s collides with another stationary ball. After the collision, the first ball moves with a velocity of 0.3 m/s in a direction making an angle of 90° with its initial direction. Calculate the momentum of second ball after collision (in kg.m/s).



Answer:
Let the second ball moves with momentum P making an angle θ from the horizontal (as shown in the figure).
By the conservation of horizontal momentum $1\times 0.4 = P\cos\theta$ ......(i)
By the conservation of vertical momentum $1\times 0.3 = P\sin\theta$ ......(ii)
From (i) and (ii) we can have
$\frac{\text{(ii)}}{\text{(i)}} \Rightarrow \tan\theta = \frac{3}{4}, \cos\theta = \frac{4}{5}, \sin\theta = \frac{3}{5},$ and no need to calculate θ

From (i) $P\cos\theta = P\left(\frac{4}{5}\right) = 0.4 \Rightarrow P = 0.5$ kg.m/s

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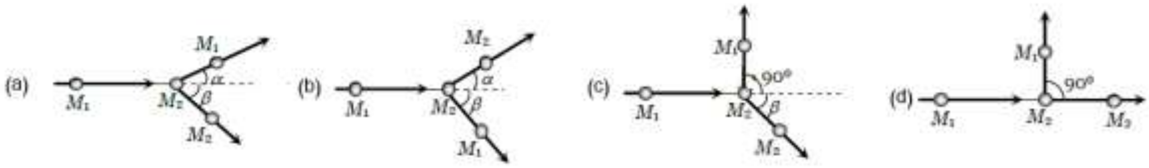
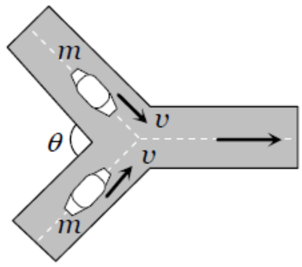
Example: Keeping the principle of conservation of momentum in mind which of the following collision diagram is not correct:



Answer: In figure (d), the final resultant momentum makes some angle with x-axis. This is not possible because initial momentum is along the x-axis and according to law of conservation of momentum initial and final momentum should be equal in magnitude and direction both.

Example: Two cars of same mass are moving with same speed v on two different roads inclined at an angle θ with each other, as shown in the figure. At the junction of these roads the two cars collide inelastically and move simultaneously with the same speed. The speed of these cars would be

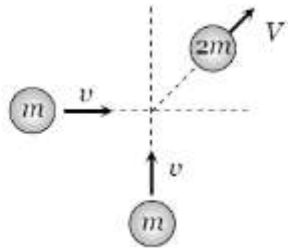


Answer: Initial horizontal momentum of the system

$$= mv\cos(\theta/2) + mv\cos(\theta/2).$$

If after the collision cars move with common velocity V then final horizontal momentum of the system = $2mV$.
By the law of conservation of momentum,

$$2mV = mv\cos(\theta/2) + mv\cos(\theta/2) \quad \Rightarrow \quad V = v\cos(\theta/2)$$



Example: A particle of mass *m* moving eastward with a speed *v* collides with another particle of the same mass moving northward with the same speed *v*. The two particles coalesce on collision. The new particle of mass 2*m* will move in the north-easterly direction with a velocity

Answer: Initially both the particles are moving perpendicular to each other with momentum *mv*. So, the net initial momentum

$$\vec{P}_i = m\vec{v}_1 + m\vec{v}_2 = \left(m|\vec{v}_1|\,\hat{\mathrm{i}} + m|\vec{v}_2|\,\hat{\mathrm{j}}\right) = m|\vec{v}|\left(m|\vec{v}|\,\hat{\mathrm{i}} + m|\vec{v}|\,\hat{\mathrm{j}}\right)$$

After the perfectly inelastic collision both the particles (system) moves with velocity *V*, so The final momentum:

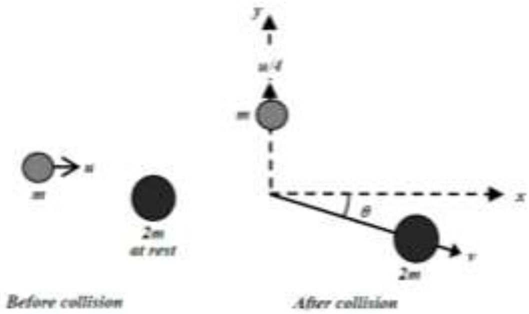
$$\vec{P}_f = 2m|\vec{V}|\left(\cos\theta\,\hat{\mathrm{i}} + \sin\theta\,\hat{\mathrm{j}}\right)$$

Conservation of momentum implies $\vec{P}_i = \vec{P}_f$ or $|\vec{P}_i|^2 = |\vec{P}_f|^2$ or $|\vec{P}_i| = |\vec{P}_f|$. The condition gives:

$$= \sqrt{(mv)^2 + (mv)^2} = \sqrt{2}mV \cdot$$

By the law of conservation of momentum $\sqrt{2}mv = 2mV \;\Rightarrow\; V = v/\sqrt{2}$.

Example: A particle of mass *m* moving in the positive *x* direction with speed *u* collides with a particle of mass *2m* at rest. After collision, the particle of mass *m* scatters with speed *u/4* in the positive *y* direction and the particle of mass *2m* moves with speed *v* making an angle *θ* with the positive *x* direction (see **Figure**). Find the angle *θ*.



Answer: Conservation of momentum $\vec{P}_i = \vec{P}_f \Rightarrow$:

x − axis:

$$\mathrm{mu} + 0 = 2\mathrm{mv}\cos\theta \;\rightarrow (1)$$

y − axis:

$$0 = \frac{\mathrm{mu}}{4} - 2\mathrm{mv}\sin\theta \;\rightarrow (2)$$

Example: Two 2.00 kg bodies, A and B, collide. Their velocities before the collision are $\vec{\mathrm{v}}_{\mathrm{Ai}} = (1.50\,\hat{\mathrm{i}} + 3.00\,\hat{\mathrm{j}})$ m/s and $\vec{\mathrm{v}}_{\mathrm{Bi}} = (1.00\,\hat{\mathrm{i}} + 0.500\,\hat{\mathrm{j}})$ m/s. After the collision, velocity of A is $\vec{\mathrm{v}}_{\mathrm{Af}} = (-0.50\,\hat{\mathrm{i}} + 2.00\,\hat{\mathrm{j}})$ m/s. What is the kinetic energy of body B after the collision?

Answer:

Initial momentum:

$$\vec{P}_i = \mathrm{m_A}\vec{\mathrm{v}}_{\mathrm{Ai}} + \mathrm{m_B}\vec{\mathrm{v}}_{\mathrm{Bi}} = \mathrm{m_A}(1.50\,\hat{\mathrm{i}} + 3.00\,\hat{\mathrm{j}}) + \mathrm{m_B}(1.00\,\hat{\mathrm{i}} + 0.500\,\hat{\mathrm{j}}) \qquad \text{(i)}$$

Final momentum:

$$\vec{P}_f = \mathrm{m_A}\vec{\mathrm{v}}_{\mathrm{Af}} + \mathrm{m_B}\vec{\mathrm{v}}_{\mathrm{Bf}} = \mathrm{m_A}(-0.50\,\hat{\mathrm{i}} + 2.00\,\hat{\mathrm{j}}) + \mathrm{m_B}\vec{\mathrm{v}}_{\mathrm{Af}} \qquad \text{(ii)}$$

Conservation of momentum $\vec{P}_i = \vec{P}_f \Rightarrow$:

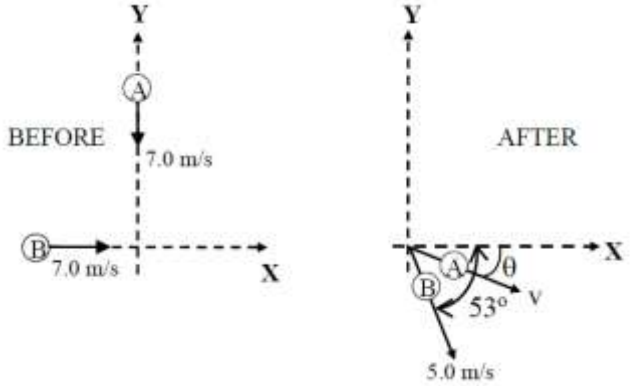
$$\vec{v}_{Bf} = \frac{m_A\vec{v}_{Ai} + m_B v_{Bi} - m_A\vec{v}_{Af}}{m_B} = \frac{m_A(\vec{v}_{Ai} - \vec{v}_{Af}) + m_B v_{Bi}}{m_B}$$

$$= \frac{\cancel{2}(1.50\,\vec{i} + 3.00\,\vec{j} + 0.5\,\vec{i} - 2.00\,\vec{j}) + \cancel{2}(-1.0\,\vec{i} + 0.5\,\vec{j})}{\cancel{2}}$$

$$= 1.01\ \vec{i} + 1.5\,\vec{j}$$

$$K_{Bf} = \frac{1}{2}m_B v_{Bf}^2 = \frac{1}{2}\times 2\times(1.0^2 + 1.5^2) = 3.25\,\mathrm{J}$$

Example: Two objects A and B, with the same mass collide on ice with negligible friction. The following **Figure** gives speeds and directions of the objects BEFORE and AFTER the collision. Find the speed v and angle θ for object A after the collision.



Answer:

along the y-axis $m_A v_A = m_A v_{Af}\sin\theta + m_B v_{Bf}\sin 53^\circ$

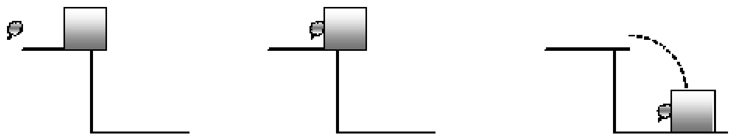
along the x-axis $m_B v_B = m_A v_{Af}\cos\theta + m_B v_{Bf}\cos 53^\circ$

Extra problems

Q: A lump of clay (m_1 = 0.2 kg) moving horizontally with a speed v_{o1} = 16 m/s strikes and sticks to a wood block (m_2 = 3 kg) which is initially at rest on the edge of a horizontal table of height h = 1.5 m. Neglecting friction, find
(a) the horizontal distance x from the edge of the table at which the clay and block strike the floor, and
(b) the total momentum of the clay and block just before they strike the floor.

Solution: The sketches below represent the system before, during, and after the collision.

Before *During* *After*



(a) Before we can find the horizontal distance the clay and block travel we need to find their speed v_f as they leave the edge of the table. Momentum is conserved in this inelastic collision:

$$\mathbf{p}_o = \mathbf{p}_f$$

$$m_1 \mathbf{v}_{o1} = (m_1 + m_2)\mathbf{v}_f$$

$$v_f = \frac{m_1 v_{o1}}{(m_1 + m_2)} = \frac{(0.2\,\text{kg})(16\frac{\text{m}}{\text{s}})}{(0.2\,\text{kg} + 3.0\,\text{kg})} = 1\frac{\text{m}}{\text{s}}$$

Now the clay and block have become a projectile which is launched horizontally. We can find the time of flight by using the height:

$$h = \frac{1}{2} g t^2$$

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(1.5\,\text{m})}{10\frac{\text{m}}{\text{s}^2}}} = 0.55\,\text{s}$$

Then the horizontal distance traveled is $x = v_f t = (1\frac{m}{s})(0.55\,s) = 0.55\,m$

(b) The momentum of the clay and block just before it strikes the ground can be found by finding the horizontal and vertical components of the momentum:

$$p_x = m v_x = (3.2)(1) = 3.2 \quad \text{kg.m/s}$$

$$p_y = m v_y = m(gt) = (3.2)(10)(0.55) = 17.6 \quad \text{kg.m/s}$$

By the Pythagorean theorem, the magnitude of the momentum can be found by

$$p = \sqrt{p_x^{\,2} + p_y^{\,2}} = \sqrt{(3.2)^2 + (17.6)^2} = 17.9\,\text{kg.m/s}$$

The angle at which the momentum is directed is

$$\theta = \tan^{-1}\left[\frac{p_y}{p_x}\right] = \tan^{-1}\left[\frac{17.6}{3.2}\right] = 80^\circ \quad \text{below the horizontal.}$$
